

Amenability in inverse semigroups and C*-algebras

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Overview

Introduction (\rightsquigarrow group case)

Inverse semigroups

Introduction

A conjecture & a problem

Amenability as domain-measurability and localization

Domain-measures as traces

Conclusions and open problems

Disclaimer

Joint work with Pere Ara (UAB) &
Fernando Lledó

Work in progress!

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4. \mathcal{R}_G is not properly infinite

($\equiv \nexists v, w \in \mathcal{R}_G$ with $1 = v^*v = w^*w \geq vv^* + ww^*$).

Inverse semigroups

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- $e, f \in E(S) \rightsquigarrow \underline{e \leq f} \Leftrightarrow ef = e$.

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Definition (Day - 1957)

S is amenable if there is a probability measure $\mu: \mathcal{P}(S) \rightarrow [0, 1]$ such that for every $s \in S$ and $A \subset S$

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Questions: *Regular representation? Paradoxical decomposition?*

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Questions:

1. When does \mathcal{R}_S have a trace?
2. When is \mathcal{R}_S properly infinite?

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$$\phi(P_A) = \lim_{\omega} \frac{|A \cap \{a^i (a^*)^j \mid i, j \leq n\}|}{|\{a^i (a^*)^j \mid i, j \leq n\}|}.$$

→ Not properly infinite.

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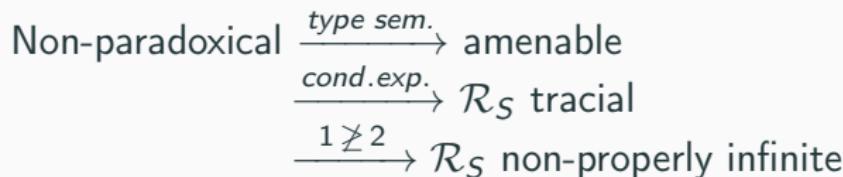
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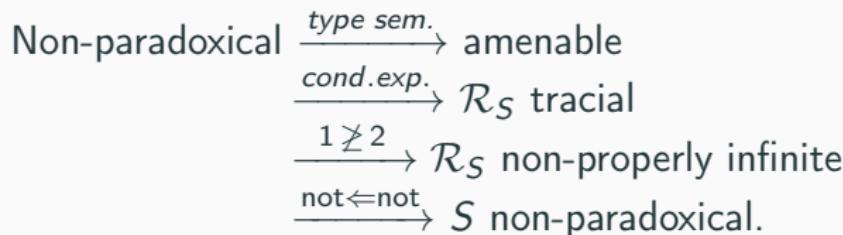
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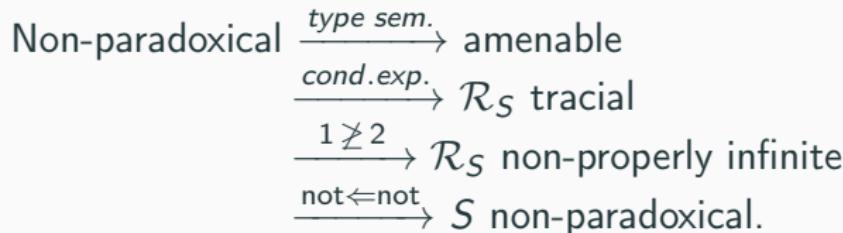
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Conjecture false... But *almost* true.

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Goal: paradox. dec. \rightsquigarrow properly infinite behaviour in $V_s \in \mathcal{R}_S$.

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Pick $|S| = \infty$ and $0 \in S$ and μ invariant ($\mu(s^{-1}A) = \mu(A)$):

$$S = 0^{-1}\{0\} \text{ while } \emptyset = 0^{-1}(S \setminus \{0\}).$$

A problem

Goal: paradox. dec. \rightsquigarrow properly infinite behaviour in $V_s \in \mathcal{R}_S$.

In groups: $\lambda_{a_i}^* P_{a_i A_i} : a_i A_i \mapsto A_i$ and these induce proper inf.

Consider: $S = a_1^{-1} A_1 \sqcup \cdots \sqcup a_n^{-1} A_n = b_1^{-1} B_1 \sqcup \cdots \sqcup b_m^{-1} B_m$
 $= A_1 \sqcup \cdots \sqcup A_n \sqcup B_1 \sqcup \cdots \sqcup B_m. (\star)$

Pick $|S| = \infty$ and $0 \in S$ and μ invariant ($\mu(s^{-1}A) = \mu(A)$):

$$S = 0^{-1}\{0\} \text{ while } \emptyset = 0^{-1}(S \setminus \{0\}).$$

A problem

(\star) does not induce properly infinite behaviour in $V_s \in \mathcal{R}_S$...

does not even induce maps $A_i \mapsto a_i^{-1} A_i \dots$

Conclusion: need a version of amenability *forwards*...

Domain-measurability and localization I

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Domain-measurable semigroups

Examples of domain-measurable semigroups:

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Definition

S is *domain-paradoxical* if there are $a_i \in S, A_i \subset S$ with

$$\begin{aligned} S &= a_1 A_1 \sqcup \cdots \sqcup a_n A_n = b_1 B_1 \sqcup \cdots \sqcup b_m B_m \\ &\supset a_1^* a_1 A_1 \sqcup \cdots \sqcup a_n^* a_n A_n \sqcup b_1^* b_1 B_1 \sqcup \cdots \sqcup b_m^* b_m B_m. \end{aligned}$$

Domain-measures as traces & Theorem I

Theorem (Ara, Lledó, M. - 2018)

S countable, discrete inverse semigroup and $1 \in S$. TFAE:

1. S is domain-measurable.
2. S is not domain-paradoxical.
3. \mathcal{R}_S has a trace.
4. \mathcal{R}_S is not properly infinite.

Key ideas of the proof:

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Key ideas of the proof:

(1) Cond. exp. $E: \mathcal{B}(\ell^2(S)) \rightarrow \ell^\infty(S)$ and $\phi = \phi \circ E$.

(2) *Domain-equidecomposability*: for $A, B \subset S$

$$A \sim B \Leftrightarrow \begin{cases} A = a_1^* a_1 A_1 \sqcup \cdots \sqcup a_n^* a_n A_n, \\ B = a_1 A_1 \sqcup \cdots \sqcup a_n A_n. \end{cases}$$

Traces and amenability

The latter argument actually gives:

$$\{\text{Traces on } \mathcal{R}_S\} \leftrightarrow \{\text{Measures on } S \mid \mu(s^* s A) = \mu(s A)\}.$$

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Furthermore, should one add *localization* to the Theorem...

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Theorem (Ara, Lledó, M. - 2018)

S is amenable

$\Leftrightarrow \mathcal{R}_S$ has a trace such that $\phi(V_{s^* s}) = 1$

\Leftrightarrow no projection $V_{s^* s} \in \mathcal{R}_S$ is properly infinite.

Conclusions and open problems

Open problems

Inverse semigroups:

1. Amenable \Rightarrow Følner \Rightarrow Domain measurable... *domain-Følner?*
2. $\{\text{Traces in } C_r^*(S)\} = \{\text{Traces in } \mathcal{R}_S\}$?
3. *Roe algebra?*

$$\mathcal{R}_S = \ell^\infty(S) \rtimes_r S$$

$$= C^*\left(\{T: \ell^2(S) \rightarrow \ell^2(S) \text{ bounded prop.}\}\right).$$

$$d_{S,K}(u, v) := \begin{cases} |[vu^*]| & \text{if } u^*u = v^*v \\ \infty & \text{otherwise} \end{cases}$$

4. Theorem based on $E: \mathcal{R}_S \rightarrow \ell^\infty(S)$ and traces factor via E ...
Other C^* -algebras whose trace space behave like this?

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Question: Results $\rightsquigarrow \dots$ groupoid context?

Bibliography & thanks

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Thank you for your attention! Questions?