Amenability in inverse semigroups and C*-algebras

Diego Martínez

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Instituto de Ciencias Matemáticas - Universidad Carlos III de Madrid
lumartin@math.uc3m.es
luisdiego.martinez@icmat.es

Overview

Introduction ($\rightsquigarrow$ group case)

Inverse semigroups

- Introduction
- A conjecture & a problem
- Amenability as domain-measurability and localization
- Domain-measures as traces

Conclusions and open problems

Disclaimer

Joint work with Pere Ara (UAB) & Fernando Lledó

Work in progress!

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4. $\mathcal{R}_G$ is not properly infinite
   $(\equiv \nexists\, v, w \in \mathcal{R}_G$ with $1 = v^*v = w^*w \geq vv^* + ww^*)$.

Inverse semigroups

Introduction to inverse semigroups I

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- $e, f \in \mathrm{E}(S) \ \rightsquigarrow \ e \leq f \Leftrightarrow ef = e$.

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Definition (Day - 1957)

S is amenable if there is a probability measure $\mu\colon \mathcal{P}(S) \to [0,1]$ such that for every $s \in S$ and $A \subset S$

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Questions: *Regular representation*? *Paradoxical decomposition*?

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Questions:

1. When does $\mathcal{R}_S$ have a trace?
2. When is $\mathcal{R}_S$ properly infinite?

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$$\phi(P_A) = \lim_\omega \frac{\left|A \cap \left\{a^i (a^*)^j \mid i,j \le n\right\}\right|}{\left|\left\{a^i (a^*)^j \mid i,j \le n\right\}\right|}.$$

$\to$ Not properly infinite.

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Conjecture false... But *almost* true.

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Consider: $S = a_1^{-1} A_1 \sqcup \cdots \sqcup a_n^{-1} A_n = b_1^{-1} B_1 \sqcup \cdots \sqcup b_m^{-1} B_m$

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$(\star)$ does <u>not</u> induce properly infinite behaviour in $V_s \in \mathcal{R}_S$...

does <u>not</u> even induce maps $A_i \mapsto a_i^{-1} A_i$...

Conclussion: *need* a version of amenbility *forwards*...

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domain-measure and *localization*.

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Definition

S is *domain-paradoxical* if there are $a_i \in S, A_i \subset S$ with

$$\begin{aligned} S &= a_1 A_1 \sqcup \cdots \sqcup a_n A_n = b_1 B_1 \sqcup \cdots \sqcup b_m B_m \\ &\supset a_1^* a_1 A_1 \sqcup \cdots \sqcup a_n^* a_n A_n \sqcup b_1^* b_1 B_1 \sqcup \cdots \sqcup b_m^* b_m B_m. \end{aligned}$$

Domain-measures as traces & Theorem I

Theorem (Ara, Lledó, M. - 2018)

S countable, discrete inverse semigroup and $1 \in S$. TFAE:

1. S is domain-measurable.
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(2) *Domain-equidecomposability*: for $A, B \subset S$

$$A \sim B \Leftrightarrow \begin{cases} A = & a_1^* a_1 A_1 \sqcup \cdots \sqcup a_n^* a_n A_n, \\ B = & a_1 A_1 \sqcup \cdots \sqcup a_n A_n. \end{cases}$$

Traces and amenability

The latter argument actually gives:

$$\{\text{Traces on } \mathcal{R}_S\} \leftrightarrow \{\text{Measures on } S \mid \mu(s^* sA) = \mu(sA)\}.$$

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Theorem (Ara, Lledó, M. - 2018)

S is amenable

$$\Leftrightarrow \mathcal{R}_S \text{ has a trace such that } \phi(V_{s^*s}) = 1$$

$$\Leftrightarrow \text{no projection } V_{s^*s} \in \mathcal{R}_S \text{ is properly infinite.}$$

Conclusions and open problems

Open problems

Inverse semigroups:

1. Amenable $\Rightarrow$ Følner $\Rightarrow$ Domain measurable... *domain-Følner*?
2. $\{\text{Traces in } C_r^*(S)\} = \{\text{Traces in } \mathcal{R}_S\}$?
3. *Roe algebra*?
$$\mathcal{R}_S = \ell^\infty(S) \rtimes_r S$$
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Question: Results $\rightsquigarrow$ ... grouopid context?

Bibliography & thanks

VON NEUMANN, Zur allgemeinen theorie des masses. Fundamenta Mathematica, 1929.

DAY, Amenable semigroups. Illinois Journal of Mathematics, 1957.

NAMIOKA, Følner's conditions for amenable semi-groups. Mathematica Scandinavica, 1965.

BARTHOLDI, On amenability of group algebras, I. Israel Journal of Mathematics, 2008.

RØRDAM AND SIERAKOWSKI, Purely infinite C^*-algebras arising from crossed products, 2010.

ARA, et al, Amenability and uniform Roe algebras. Journal of Mathematical Analysis and Applications, 2018.

ARA, LLEDÓ AND M., Amenability in semigroups and C^*-algebras, preprint, 2018.

Thank you for your attention! Questions?