Coarse Geometry and Inverse Semigroups

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Overview

(1) Crash Course on Coarse Geometry

Shifting the point of view of geometry & examples

Properties *at infinity*

More examples?

(2) Inverse semigroups

Definition & examples

The *Cayley* graph of an inverse semigroup

Amenability in inverse semigroups

(3) Relation with Operator Algebras

Introduction to uniform Roe algebras

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Recall: usual approach to (differential) geometry:

- Manifolds $\rightsquigarrow$ object of study.

An introduction to coarse geometry

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(X, d_X), (Y, d_Y). Then: $\phi : X \to Y$ is a quasi-isometry if:

(1) There are $L, C > 0$ such that

$$\frac{1}{L} d_X(x, x') - C \leq d_Y(\phi(x), \phi(x')) \leq L d_X(x, x') + C.$$

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(2) $d_Y(y, \phi(X)) \le R$ for some $R > 0$ and all $y \in Y$.

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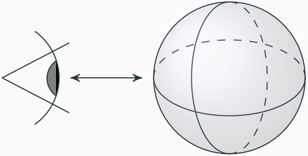

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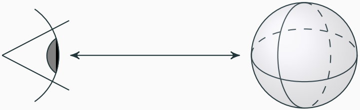

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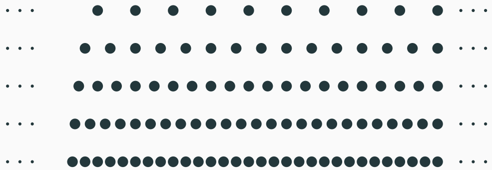

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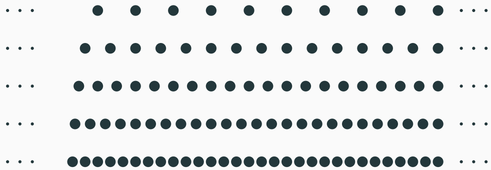

Facts: from the above picture it follows that:

- $\mathrm{id}\colon \mathbb{Z} \to \mathbb{R}$ is a quasi-isometry.
- $\mathbb{R}$ and $\mathbb{R}^2$ are <u>not</u> quasi-isometric.
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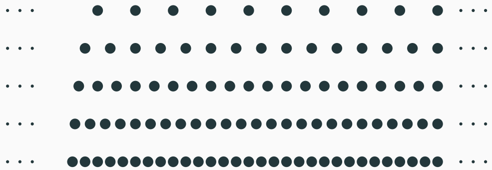

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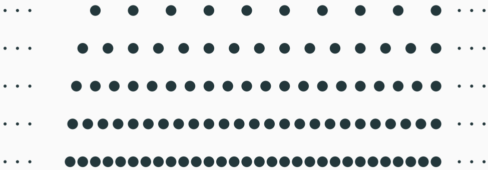

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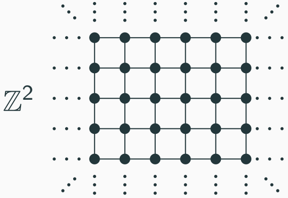

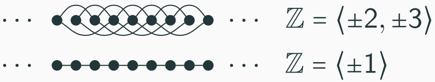

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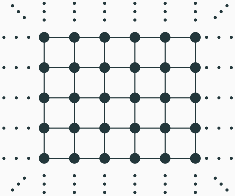
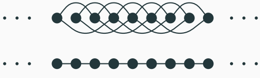
$\mathbb{Z}^2$





$\mathbb{Z} = \langle \pm 2, \pm 3 \rangle$

$\mathbb{Z} = \langle \pm 1 \rangle$

Proposition

The large scale geometry of the Cayley graph of G does not depend on the generators, i.e.,

$$\mathrm{Cay}\,(G, \{g_1, \ldots, g_n\}) \cong_{\text{q.i.}} \mathrm{Cay}\,(G, \{h_1, \ldots, h_m\}).$$

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For the rest: metric space is $\text{Cay}(G)$, G finitely generated.

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Definition (von Neumann '29 & Følner '57)

G is amenable if there is $\{F_n\}_{n\in\mathbb{N}}$, where $\varnothing \neq F_n \subset\subset G$ and

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Proof: Følner sets are preserved under quasi-isometry.

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Definition (Yu '99)

G has <u>A</u> if for all $\varepsilon, R > 0$ there is $\xi\colon G \to \ell^1(G)$ such that

1. $\xi_g \geq 0$ and $\|\xi_g\|_1 = 1$ for all $g \in G$.
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Or, rather, is a *global* property.

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Answer: Using semigroups we can.

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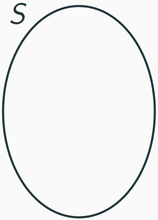

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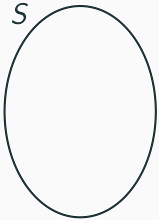

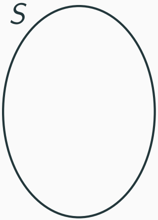

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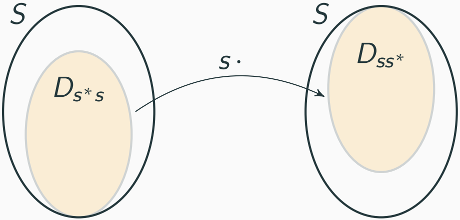

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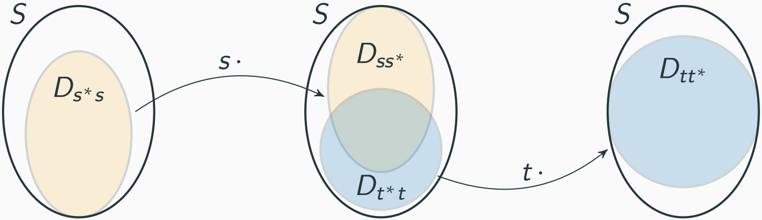

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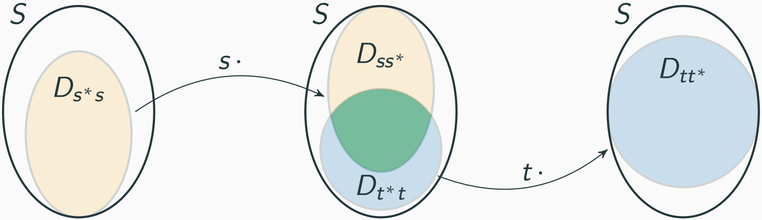

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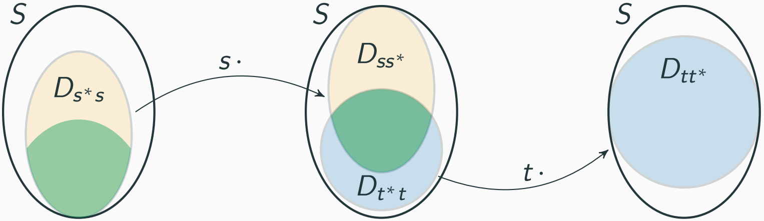

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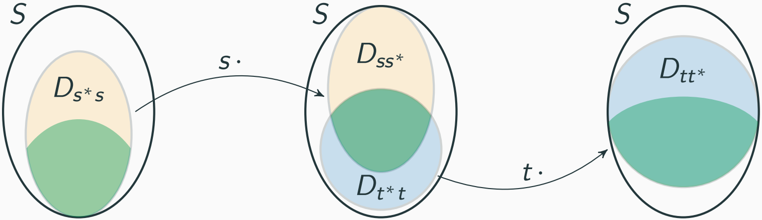

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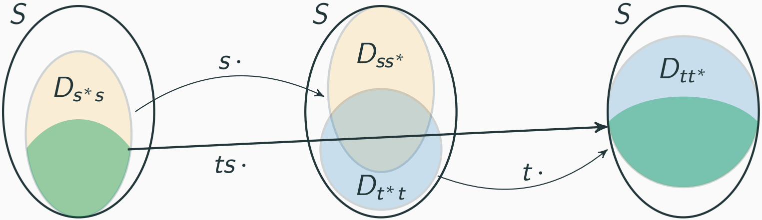

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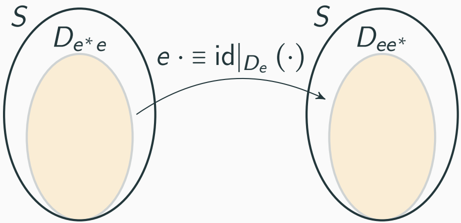

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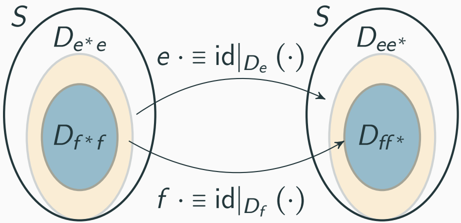

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\vdots & \vdots & \vdots & \vdots & \iddots \\
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Definition

Let $S = \langle\, s_1, \ldots, s_n \mid \text{relations}\,\rangle$, and $u, v \in S$. Then:

$$d(u, v) := \begin{cases} \min\{\ell(s) \mid su = v \text{ and } s^* su = u\} & \text{if} \quad u^* u = v^* v, \\ 0 & \text{if} \quad u^* u \neq v^* v. \end{cases}$$

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- $\mathcal{C} := \{$cycles starting and ending at $v_0\} = \{\, w^* w \mid w \in V \}$.
- $S = \left\langle\, \{w, w^*\}_{w \in V} \mid \{\, c = v_0 \,\}_{c \in \mathcal{C}} \text{ and } v_0 = v_0^* v_0 \,\right\rangle$.

Amenability in inverse semigroups

Def. (Day - 1957) & Prop. (Ara, Lledó, M. - 2019)

S is amenable if there is an invariant probability measure on it:
a probability measure $\mu: \mathcal{P}(S) \to [0,1]$ such that

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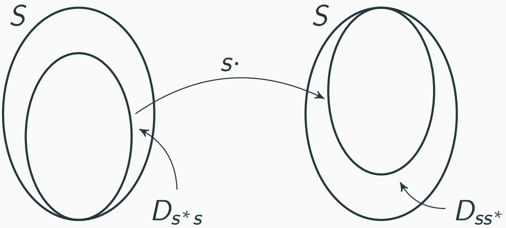

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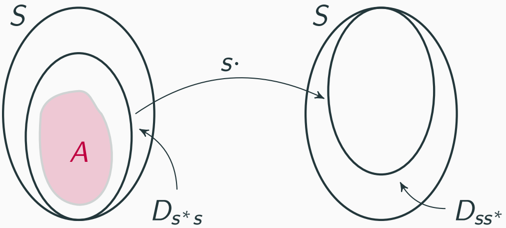

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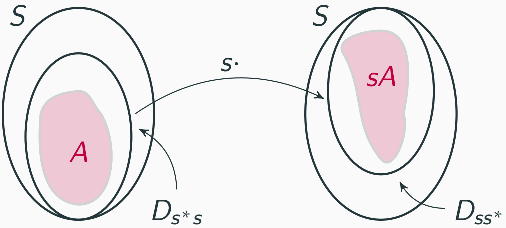

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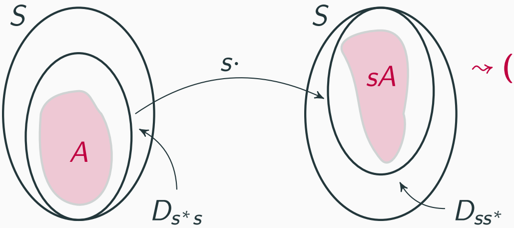

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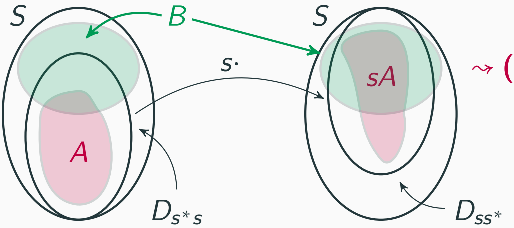

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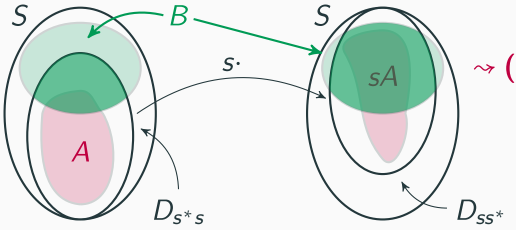

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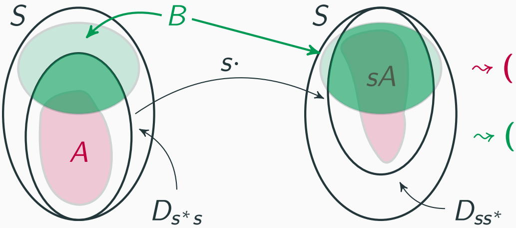

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Remark: localization is not dynamical... But it allows:

- Amenability of S can be cut down to *amenability* of D_{s^*s}.
- Amenability $\Leftrightarrow$ Følner property from within D_{s^*s}.
- Følner property $\rightsquigarrow$ amenability is a quasi-isometry invariant.

Amenability vs. large scale geometry and A

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Study when uniform Roe algebra $\cong \ell^\infty(S) \rtimes_r S$.

(3) Relation with Operator Algebras

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Observe:

- $\ell^\infty(G)$ allows to restrict to subets of G.
- Action of G allows to move points in G to points in G.
  by multiplying on the left.

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Introduction to uniform Roe algebras

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Theorem

For a discrete and countable group G:

$$\overline{\text{span}\{\text{bijections in } G \text{ coarsely preserving distances}\}} = \\ = \ell^\infty(G) \rtimes_G = \mathcal{R}_G.$$

Uniform Roe algebras of inverse semigroups

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S inverse semigroup, countable and discrete. In some cases:

$$\mathcal{R}_S \cong \overline{\text{span}\{\text{bijections in } (S, d) \text{ coarsely preserving distances}\}}.$$

Bibliography & thanks

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Thank you for your attention! Questions?