Non group-like metrics appearing in group-like objects

Diego Martínez

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Instituto de Ciencias Matemáticas - Universidad Carlos III de Madrid
lumartin@math.uc3m.es
UK Operator Algebras Seminar

Groups and their geometries

G $\rightsquigarrow$ countable and discrete group

Proposition

G is equipped with a unique (up to *c.e.*) proper and right invariant metric: $d(g,h) = d(gx, hx)$ for every $g, h, x \in G$

Sketch: Choose r-balls $\Gamma_1 \subset \Gamma_2 \subset \cdots \subset G$ with $\Gamma_r = \Gamma_r^{-1}$

$$d(g,h) = \min\left\{r \in \mathbb{N} \mid gh^{-1} \in \Gamma_r\right\}$$

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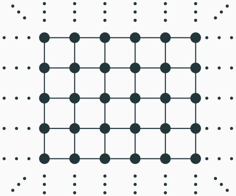
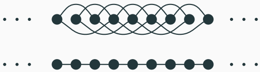
$\mathbb{Z}^2$





$\mathbb{Z} = \langle \pm 2, \pm 3 \rangle$

$\mathbb{Z} = \langle \pm 1 \rangle$

Non-group geometries and inverse semigroups

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$$
\begin{tikzcd}[column sep=small]
\vdots & \vdots & \vdots & \vdots & \iddots \\
a^3 \arrow[d, dash] & a^3a^* \arrow[d, dash] & a^3a^{*2} \arrow[d, dash] & a^3a^{*3} \arrow[d, dash] & \dots \\
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Key: if $s^*s \neq t^*t$ then $d(s,t) = \infty$ (this is *unavoidable*!)

Coarse invariants

Amenability: expressed algebraically, and is *almost* a coarse inv:

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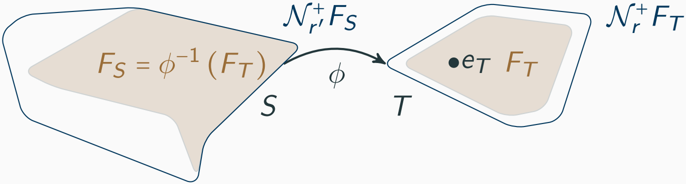

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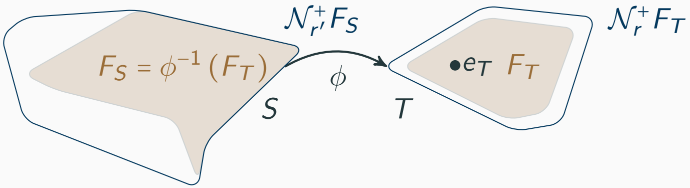

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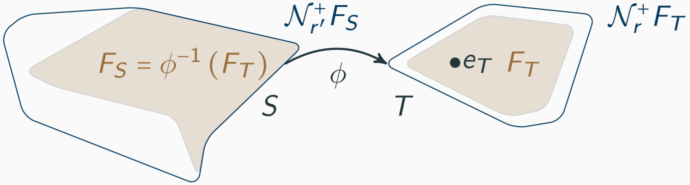

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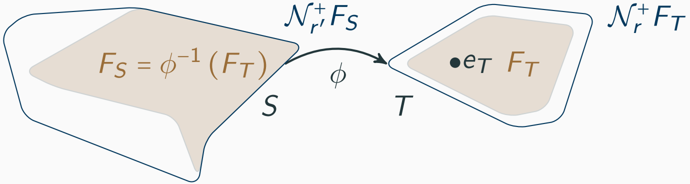

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Theorem (Lledó, M.)

Let S be an inverse semigroup. *In some cases*:

$$S \text{ has A} \Leftrightarrow C^*_r(S) \text{ is exact} \Leftrightarrow \ell^\infty(S) \rtimes_r S \text{ is nuclear}$$