Some quasi-isometric invariants for inverse semigroups

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Overview

(1) Coarse geometry and properties *at infinity*

(2) Inverse semigroups

Schützenberger graphs of an inverse semigroup

(3) Quasi-isometric invariants: amenability

(4) Quasi-isometric invariants: Yu's property A

(1) Coarse geometry and properties *at infinity*

Crash course on coarse geometry

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Quasi-definition

(X, d_X), (Y, d_Y) and $\phi : X \to Y$. We say ϕ is a:

- A *quasi-embedding* if there are $L, C > 0$ such that

$$\frac{1}{L} d_X(x, x') - C \leq d_Y(\phi(x), \phi(x')) \leq L d_X(x, x') + C.$$

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- *Quasi-surjective* if for some $R > 0$ and all $y \in Y$
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- A *quasi-isometry* when ϕ is a quasi-surjective quasi-embedding.

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Examplest

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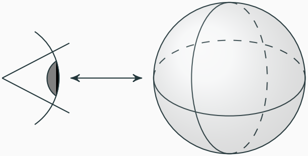

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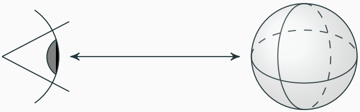

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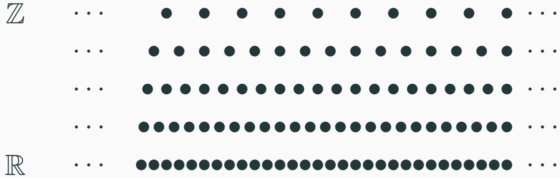

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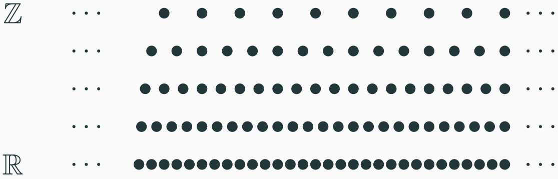

Facts: from the above picture it follows that:

- id: $\mathbb{Z} \to \mathbb{R}$ is a quasi-isometry.
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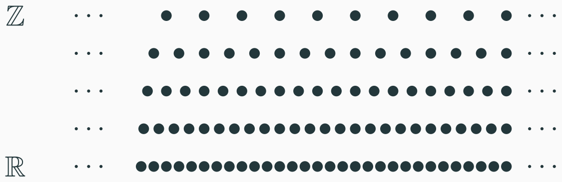

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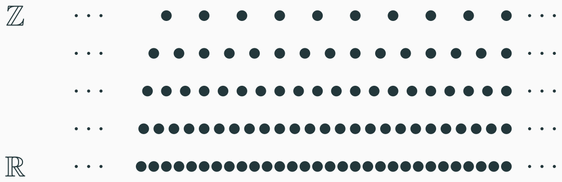

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Question: any other natural examples? **Answer:** Yes.

A factory of examples

Recall: Cayley graph construction $\rightsquigarrow G = \langle\, g_1^{\pm 1}, \ldots, g_n^{\pm 1} \,|\, \text{relations} \,\rangle$:

Graph $\rightsquigarrow \mathrm{Cay}\,(G, \{g_1, \ldots, g_n\}) := (V, E)$, where

$V := G \quad$ and $\quad E := \left\{ \left(x, x g_i^{\pm 1}\right) \mid x \in G \text{ and } i = 1, \ldots, n \right\}.$

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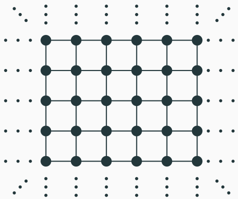

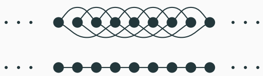

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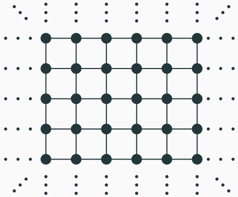

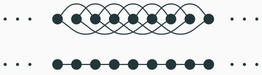

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Proposition

The large scale geometry of the Cayley graph of G

does <u>not</u> depend on the generators, i.e.,

$$\mathrm{Cay}\,(G, \{g_1, \ldots, g_n\}) \cong_{\text{q.i.}} \mathrm{Cay}\,(G, \{h_1, \ldots, h_m\})$$

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Observe: The half line $\mathbb{N}$ is not a group Cayley graph.

Therefore: We're missing quite a lot of graphs.

(2) Inverse semigroups

Brief introduction to inverse semigroups

Disclaimer: operator algebraist $\rightsquigarrow$ $^{-1}$ vs *

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Resulting object: undirected graph.

- Connected components $\rightsquigarrow$ *Schützenberger graphs*.
- In particular: $d(s,t) = d(t,s)$
- If $xx^{-1} \neq yy^{-1}$ then $d(x,y) = \infty$

Schützenberger graphs

Examplest: Bicyclic monoid: $\mathcal{T} := \langle a, a^{-1} \mid a^{-1}a = 1 \rangle$:

$$
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Proposition [Gray-Kambites (2013)]

The large scale geometry of the Schützenberger graph of S does not depend on the generators, i.e.,

$$(S, d, \{s_1, \ldots, s_n\}) \cong_{\text{q.i.}} (S, d', \{t_1, \ldots, t_m\})$$

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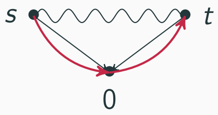

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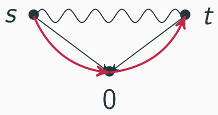

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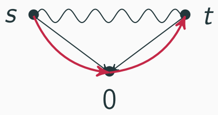

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Reason #2: *something something C^*-algebras something*

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Theorem [Stephen, 1990]

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- $\mathcal{R}$-classes in a $\mathcal{D}$-class are isomorphic (as graphs).
- ${}^{-1}\colon S \to S$ takes $\mathcal{R}$-classes $\rightsquigarrow$ $\mathcal{L}$-classes

  and, thus: Left approach $\cong_{q.i.}$ Right approach

(3) Quasi-isometric invariants: amenability

Amenability in inverse semigroups

Def. (Day - 1957) & Prop. (Ara, Lledó, M. - 2019)

S is amenable if there is an invariant probability measure on it: a probability measure $\mu : \mathcal{P}(S) \to [0,1]$ such that

(1) Domain-measure: $\mu(A) = \mu\left(As^{-1}\right)$ for all $A \subset S \cdot s^{-1}s$.

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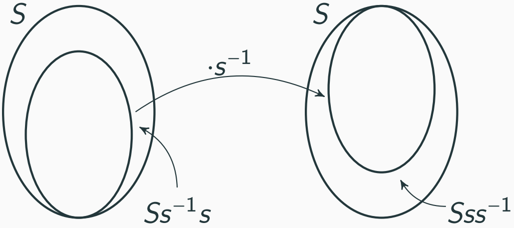

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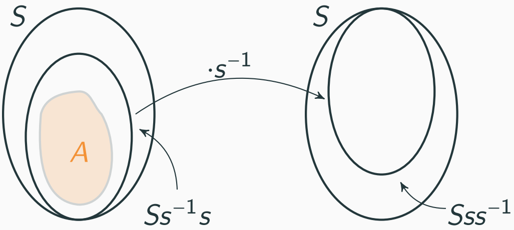

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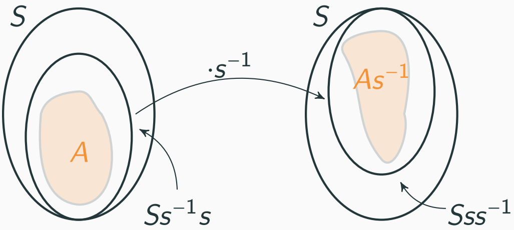

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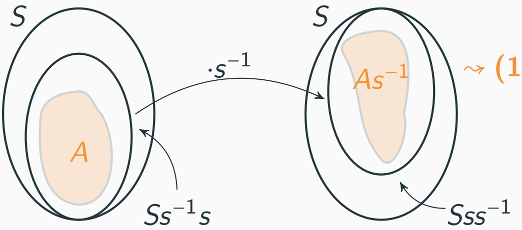

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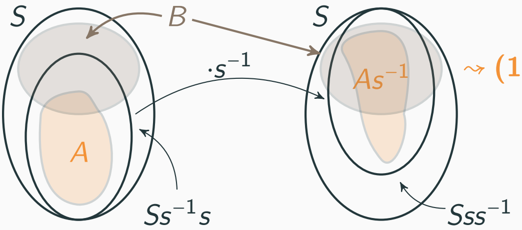

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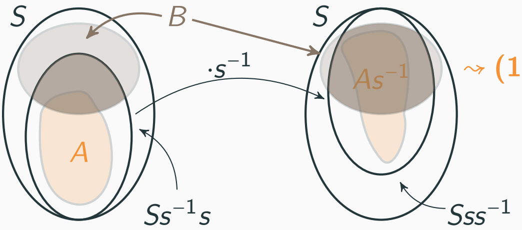

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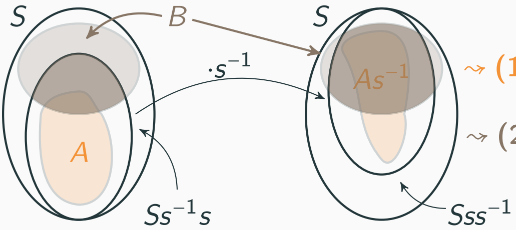
(1) Domain-measure: $\mu(A) = \mu\left(As^{-1}\right)$ for all $A \subset S \cdot s^{-1}s$.

(2) Localization: $\mu\left(S \cdot s^{-1}s\right) = 1$ for all $s \in S$.

Invariance *explanation*:



$\rightsquigarrow$ **(1)** $\mu(A) = \mu\left(As^{-1}\right)$

$\rightsquigarrow$ **(2)** $\mu(B) = \mu\left(B \cap Ss^{-1}s\right)$

$= \mu\left(B \cap Sss^{-1}\right)$

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Theorem (Ara, Lledó, M. - 2019)

S is domain-measurable iff $\exists \{F_n\}_{n\in\mathbb{N}}, \varnothing \neq F_n \subset S$ finite and

$$\left|\left(F_n \cap S \cdot s^{-1}s\right) s^{-1} \cup F_n\right| / |F_n| \xrightarrow{n\to\infty} 1 \quad \text{for every } s \in S.$$

Domain-measurability in the Schützenberger graphs

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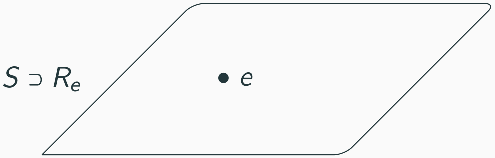

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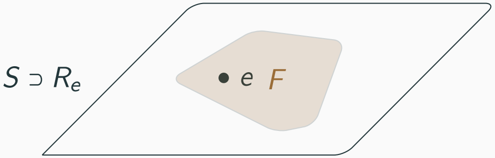

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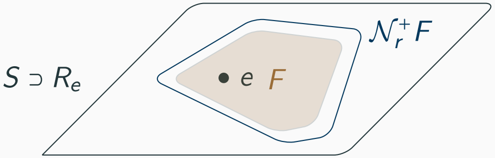

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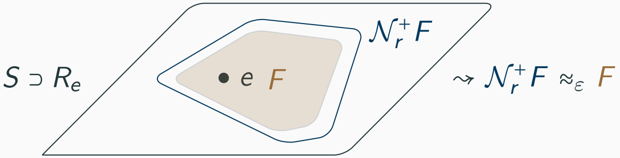

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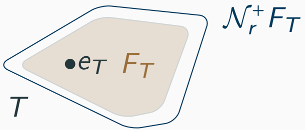

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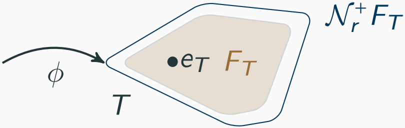

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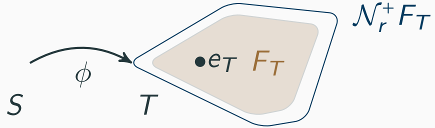

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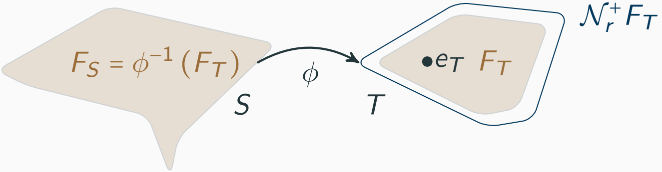

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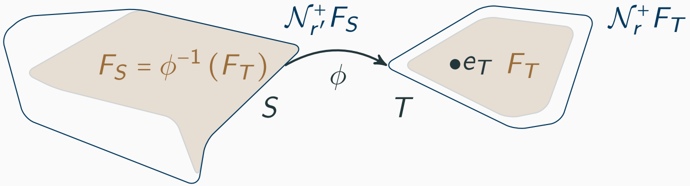

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Then: ϕ is a q.i., S non-amenable and T amenable.

domain-measurability preserved by q.i.

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amenability not preserved by q.i.

(4) Quasi-isometric invariants: Yu's property A

Definition of A

Definition (Yu - 1999)

(X, d) has *property A* if for all $\varepsilon > 0$ and $R > 0$ there are $C > 0$ and $\zeta : X \to \ell^1(X)^1_+$ with:

1. Controlled support: $\operatorname{supp}(\zeta_x) \subset B_C(x)$ for every $x \in X$.
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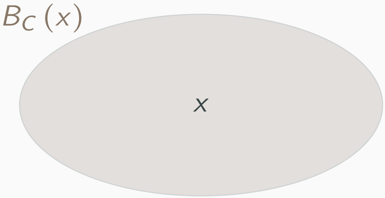

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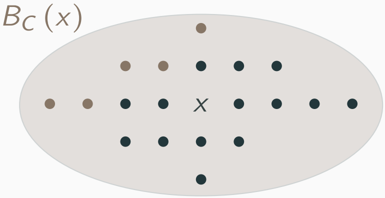

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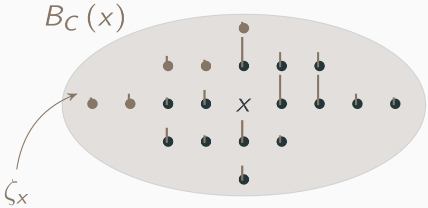

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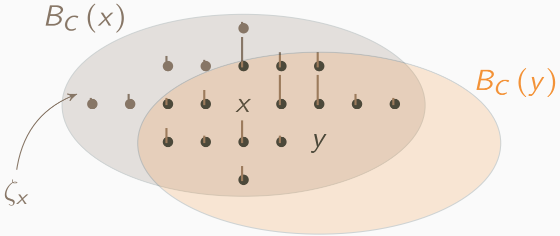

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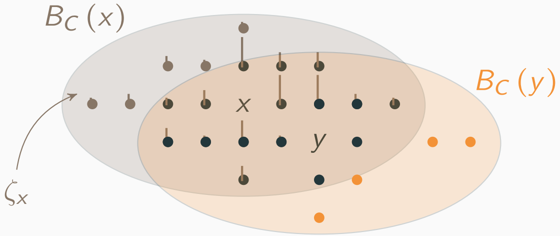

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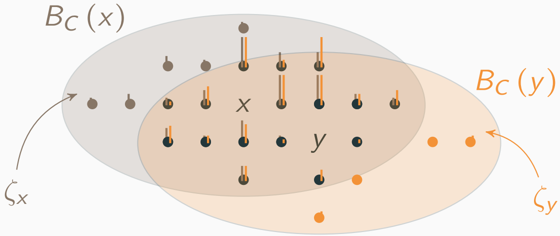

Property A for groups and its' relevance

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Therefore: any method to

- Say when a group has A
- A metric space/group does not have A

is interesting.

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Question #1: When does R have A? (For an $\mathcal{R}$-class).

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S has A $\Rightarrow\ \zeta_{[x]}([y]) := \lim_{e\searrow E} \zeta_{ex}(ey)$.

Lifting A of G via some C*-nonsense.

Question #1: When does R have A? (For an $\mathcal{R}$-class).

Question #2: When does S have A? (For $S = \sqcup_{e \in E} R_e$).

Lemma

S has A $\Leftrightarrow \{R_e\}_{e \in A}$ is *uniformly A* (i.e. $\sup_{e \in E} C_e < \infty$)

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Lifting A of G via some C*-nonsense.

Theorem (Lledó, M. - 2020)

If S is E-unitary, then S has A if and only if G has A.

Schützenberger graphs and A II

Example: Higson-Lafforgue-Skandalis and Willett:

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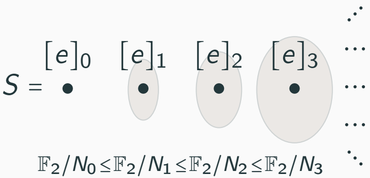

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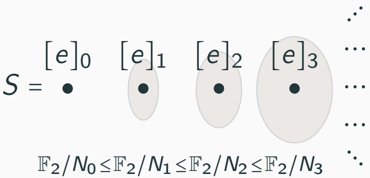

Conclusion: two reasons for S not being A:

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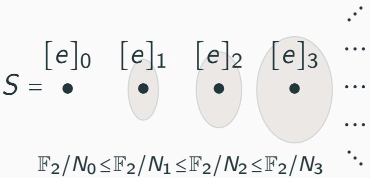

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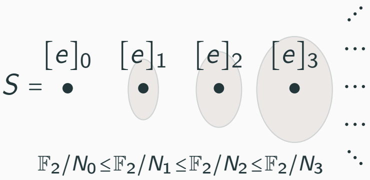

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Reason #2: $\{R_e\}_{e\in E}$ is not uniformly A.

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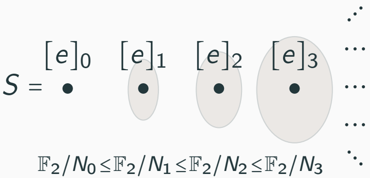

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Thank you for your attention! Questions?

About the alternative distance II

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Reason #3: *something something C^*-algebras something*