C* and geometric properties of inverse semigroups

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Noncommutative geometry seminar - NYC

Based on joint work with Pere Ara and Fernando Lledó

Outline

(1) Inverse semigroups

(2) Schützenberger graphs and right invariance

(3) Day's amenability vs. nuclearity

(4) Property A vs. exactness

(5) *Having finite local structure*

1. Inverse semigroups

Inverse semigroups and Wagner-Preston

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- $\underline{D_{s^*s} := \{ x \in S \mid x = s^*sx \}} = s^*s \cdot S$ is the *domain of s*
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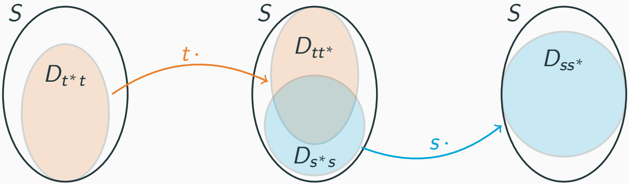
Induces the *Wagner-Preston* representation $v \colon S \to \mathcal{I}(S)$:



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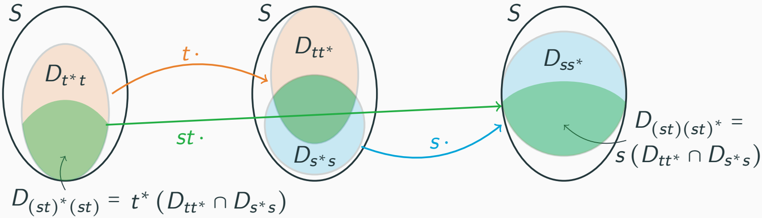

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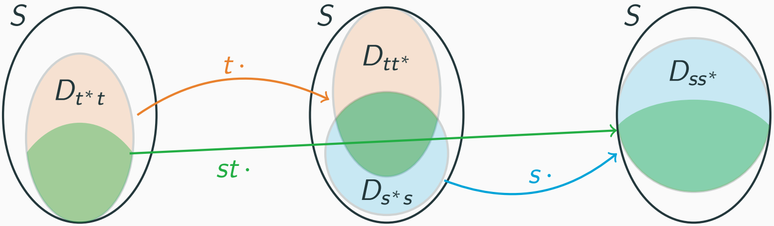

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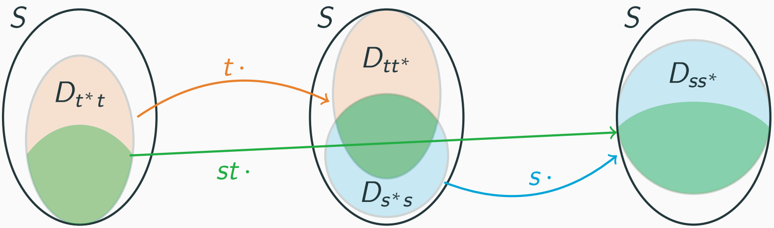

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$\Leftrightarrow$ t is a *restriction* of $s \Leftrightarrow st^*t = t$

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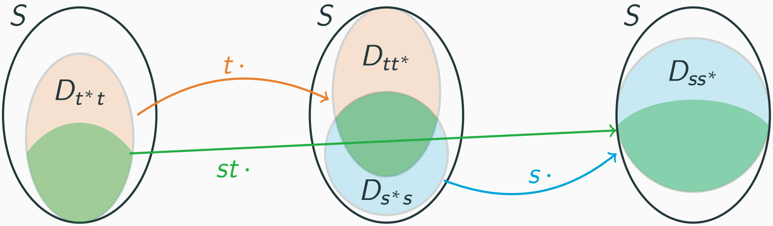

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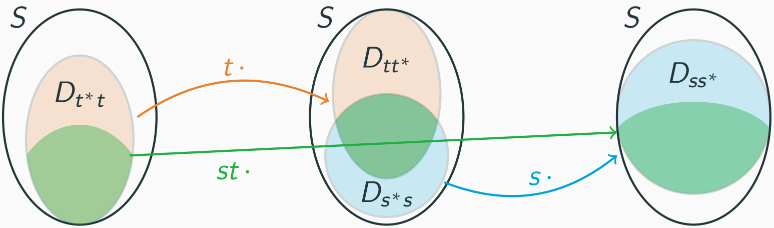

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Reduced C*-algebra: $C_r^*(S) := C^*\left(\{v_s\}_{s\in S}\right) \quad \subset \mathcal{B}\left(\ell^2(S)\right)$

Uniform Roe algebra: $\mathcal{R}_S := C^*\left(\{fv_s\}_{s\in S, f\in\ell^\infty(S)}\right) \quad \subset \mathcal{B}\left(\ell^2(S)\right)$

Recall: Cayley graph construction $\rightsquigarrow G = \langle\, g_1^{\pm 1}, \ldots, g_n^{\pm 1} \,|\, \text{relations} \,\rangle$:

- Graph $\rightsquigarrow \mathrm{Cay}\,(G, \{g_1, \ldots, g_n\}) := (V, E)$,
- Vertices $\rightsquigarrow V := G$
- Edges $\rightsquigarrow E := \left\{ \left(x, g_i^{\pm 1} x\right) \mid x \in G \text{ and } i = 1, \ldots, n \right\}$.

Group coarse geometry

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Proposition (classical)

The large scale geometry of the Cayley graph of G
does **not** depend on the generators

Goal: coarse geometry of inverse semigroups
and its relation with C*-properties of $C_r^*(S)$ and $\mathcal{R}_S$

2. Schützenberger graphs and right invariance

Infinite distances, and why they are necessary

Remark: we need to consider *extended* metric spaces:

$$\text{if } \; x \in D_{s^*s} = \{y \in S \mid s^* s y = y\} \; \text{ then } \; (sx)^* (sx) = x^* x$$

and hence $x \mathcal{L} sx$ (the converse also holds)

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Green's relations:

- $x \mathcal{L} y$ if $x^*x = y^*y$
- $x \mathcal{R} y$ if $xx^* = yy^*$
- $\mathcal{H} = \mathcal{L}$ and $\mathcal{R}$
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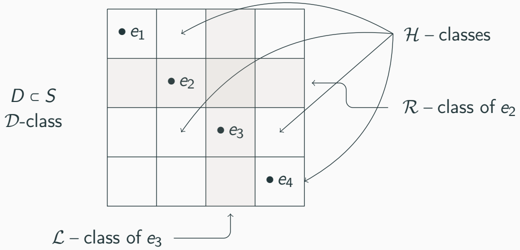

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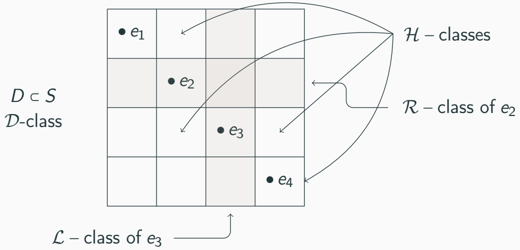

Remark - need of extended metrics

Good distances $d \colon S \times S \to [0, \infty]$ satisfy that

$$x^* x = y^* y \Leftrightarrow d(x, y) < \infty$$

Schützenberger graphs I: $\mathcal{L}$-classes

Definition (Schützenberger - 1959)

Let $S = \langle K \rangle$, where $K = K^*$. Given an $\mathcal{L}$-class $L \subset S$, let Λ_L be

- the graph whose vertices are the points of L and
- where $x, y \in L$ are joined by a k-labeled edge if $kx = y$.

Likewise, let $\Lambda_S = \sqcup_{e \in E(S)} \Lambda_{L_e}$.

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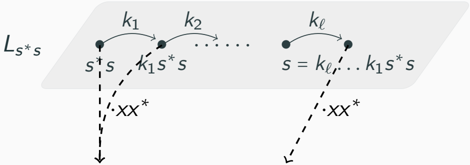

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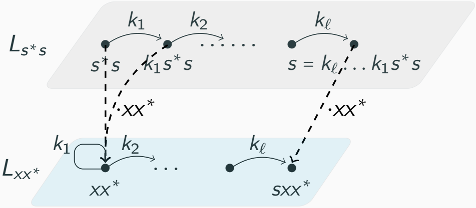

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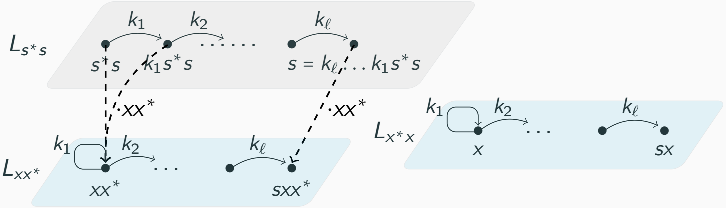

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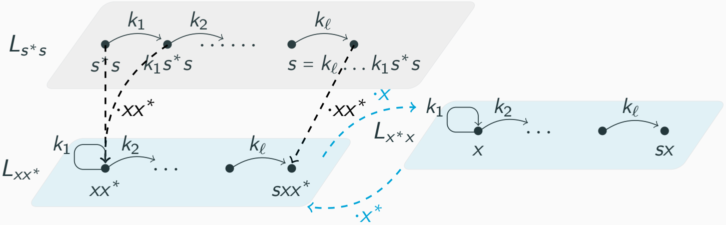

3. Day's amenability vs. nuclearity

Day's amenability as a coarse condition

Definition & Proposition (Day 1957 & Ara-Lledó-M. 2020)

S is *amenable* if there is $\mu : \mathcal{P}(S) \to [0,1]$ such that

(a) $\mu(D_{s^*s}) = 1$ for every $s \in S$
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Theorem (Ara-Lledó-M. 2020)

S is domain-measurable $\Leftrightarrow$ there is $e \in E$ and $F_n \subset L_e$ such that

$$\frac{|s(F_n \cap D_{s^*s}) \cup F_n|}{|F_n|} \xrightarrow{n\to\infty} 1 \quad \text{for all } s \in S$$

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Theorem (Ara-Lledó-M. 2020)

Let S be a unital inverse semigroup. TFAE:

(i) S is domain-measurable.
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WARNING: Day's amenability is not related with nuclearity!

Examples:

- $\mathbb{F}_2 \sqcup \{0\}$ is Day's amenable but not nuclear
- An example of Nica is nuclear but not Day's amenable

4. Property A vs. exactness

Schützenberger graphs and property A

Definition (Yu - 1999)

(X, d) has *property A* if for every $r, \varepsilon > 0$ there is

$$\xi\colon X \to \ell^1(X)_1^+ \text{ and } c > 0 \text{ such that } \operatorname{supp}(\xi_x) \subset B_c(x) \text{ and}$$

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Remarks:

- Property A generalizes amenability <u>for groups</u> (**not** in general)
- Non-property A groups are hard to come by

Theorem (Ozawa - 2000)

Let G be a countable group. The following are equivalent:

(1) G has property A.
(2) $\mathcal{R}_G$ is nuclear.
(3) $C_r^*(G)$ is exact.

Property A, nuclearity and exactness for inverse semigroups

Theorem (Lledó, M. - 2021)

Let $S = \langle K \rangle$ be an inverse semigroup. *Some* times, TFAE:

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Proof: (i) $\Rightarrow$ (ii) given $\xi\colon S \to \ell^1(S)_1^+$ the diagram

$$\begin{array}{ccc} \mathcal{R}_S \to \prod_{x\in S} M_{B_c(x)} & \subset & \ell^\infty(S) \otimes M_q \to \mathcal{R}_S \\ a \mapsto \left(p_{B_c(x)}\, a\, p_{B_c(x)}\right)_{x\in S} & \rightsquigarrow & (b_x)_{x\in S} \mapsto \sum_{x\in S} \xi_x^* b_x \xi_x \end{array}$$

can be shown to be an approximation of $\mathrm{id}\colon \mathcal{R}_S \to \mathcal{R}_S$

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Question: some times? When is that?

5. Having finitely complex local structures

Finite labeling I: local structure

Note: Λ_S = vertices + edges, and edges represent (x, sx)

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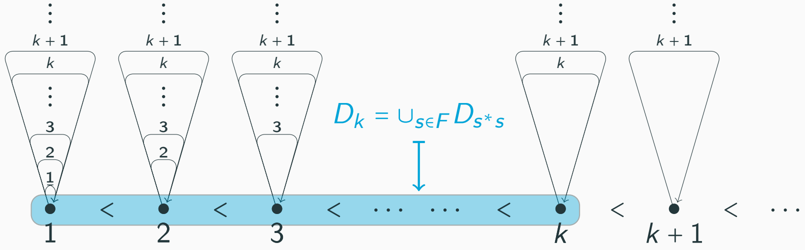

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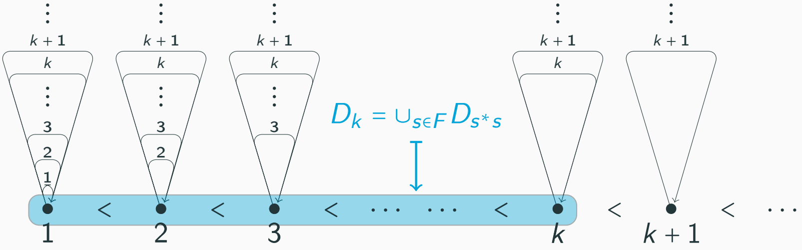

Key: for large $r \geq 0$ $\nexists$ $F \Subset S$ labeling the paths in Λ_S of length r

Finite labeling II: a picture to *top* the explanation

Definition (Lledó, M. - 2021)

Let $S = \langle K \rangle$. We say (S, K) admits a *finite labeling* if

for any $r \geq 0$ there is $F \Subset S$ such that if $d(s^*s, s) \leq r$

then there is $m \in F$ such that $ms^*s = s$.

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(1) Finitely generated semigroups
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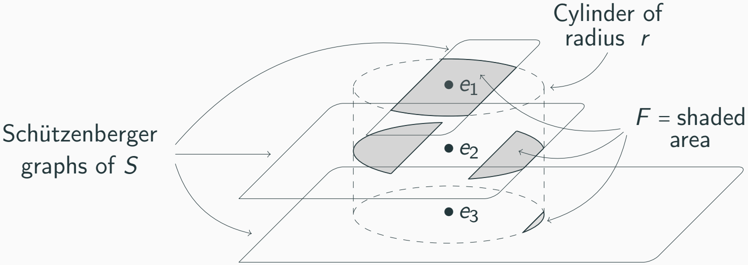

Finite labeling III: importance

Theorem (Lledó, M. - 2021)

Suppose that (S, K) admits a finite labeling. Then:

$$\mathcal{R}_S \cong \ell^\infty(S) \rtimes_r S \cong C^*_u(\Lambda_S)$$

for a certain canonical action $S \curvearrowright \ell^\infty(S)$.

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Actually, even more is true:

Theorem (Lledó, M. - 2021)

$$\mathcal{R}_S \cong C_u^*(\Lambda_S) \iff (S, K) \text{ admits a FL.}$$

Finite labelings characterizing the uniform Roe algebra

Theorem (Lledó, M. - 2021)

$$\mathcal{R}_S \cong C_u^*(\Lambda_S) \iff (S,K) \text{ admits a FL.}$$

Recall $C_u^*(\Lambda_S) = C^*\left(\left\{t \in \mathcal{B}\left(\ell^2(S)\right) \text{ of finite propagation }\right\}\right)$

Proof:

- If t has finite propagation $+$ (S,K) admits a FL
  then $\rightsquigarrow$ $t = \sum_{s\in F} f_s v_s$, where $f_s \in \ell^\infty(S)$
  i.e., we can label the pairs (x,y) with $\langle \delta_y, t\delta_x\rangle \neq 0$ as pairs (x, sx) with $s \in F \Subset S$. Thus $\mathcal{R}_S \supset C_u^*(\Lambda_S)$
- If not FL $\rightsquigarrow$ construct t of propagation 1 with $t \notin \mathcal{R}_S$

Conclusions and future work

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- Inverse semigroups can be seen geometrically

  Coarse geometry $\leftrightsquigarrow$ C*-properties
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Thank you for your attention!

Questions?