Inverse semigroups, their non-group-like geometries, and their amenability and property A

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Web seminar of operator algebras - Universidade Federal de Santa Catarina

Based on joint work with Pere Ara and Fernando Lledó

Outline

(1) Inverse semigroups
(2) Schützenberger graphs and right invariance
(3) Day's amenability vs. nuclearity
(4) Property A vs. exactness
(5) *Having finite local structure*

1. Inverse semigroups

Inverse semigroups and Wagner-Preston

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- $\underline{D_{s^*s} := \{ x \in S \mid x = s^*sx \}} = s^*s \cdot S$ is the *domain of s*
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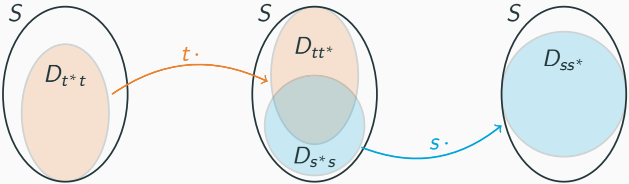

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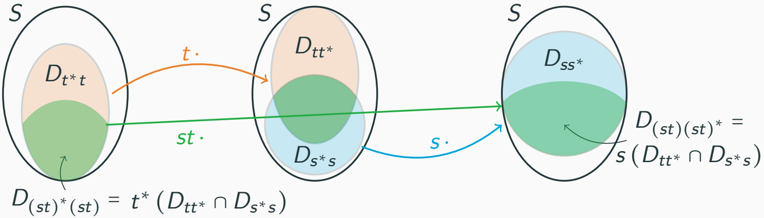

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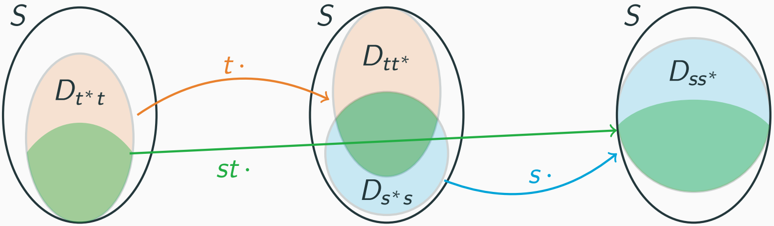

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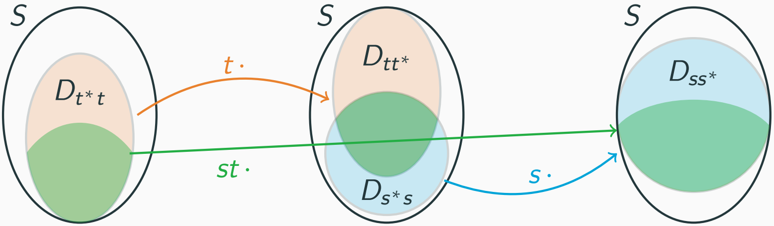

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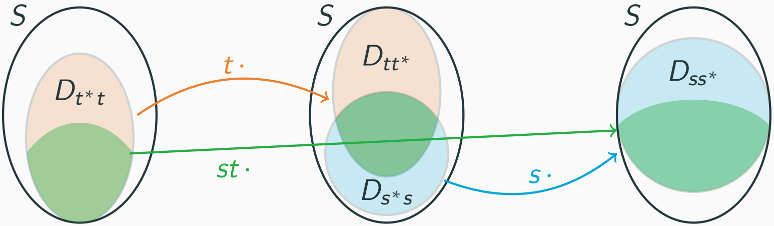

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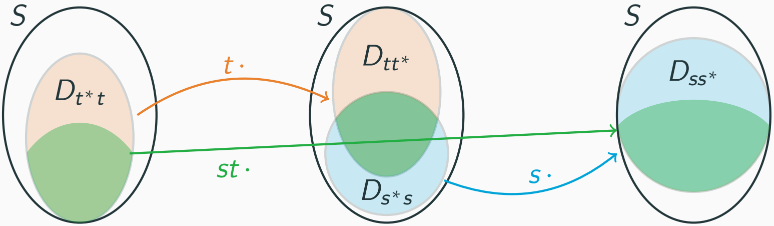

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<u>**Reduced C*-algebra:**</u> $C_r^*(S) := C^*\left(\{v_s\}_{s\in S}\right) \quad \subset \mathcal{B}\left(\ell^2(S)\right)$

<u>**Uniform Roe algebra:**</u> $\mathcal{R}_S := C^*\left(\{fv_s\}_{s\in S, f\in\ell^\infty(S)}\right) \quad \subset \mathcal{B}\left(\ell^2(S)\right)$

Group coarse geometry

Recall: Cayley graph construction $\rightsquigarrow G = \langle\, g_1^{\pm 1}, \ldots, g_n^{\pm 1} \,|\, \text{relations} \,\rangle$:

- Graph $\rightsquigarrow \mathrm{Cay}\,(G, \{g_1, \ldots, g_n\}) := (V, E)$,
- Vertices $\rightsquigarrow V := G$
- Edges $\rightsquigarrow E := \big\{\big(x, g_i^{\pm 1} x\big) \mid x \in G \text{ and } i = 1, \ldots, n\big\}$.

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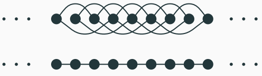

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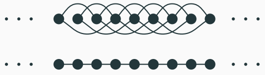

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Proposition (classical)

The large scale geometry of the Cayley graph of G
does **not** depend on the generators

Goal: coarse geometry of inverse semigroups
and its relation with C*-properties of $C_r^*(S)$ and $\mathcal{R}_S$

2. Schützenberger graphs and right invariance

Infinite distances, and why they are necessary

Remark: we need to consider *extended* metric spaces:

$$\text{if } x \in D_{s^*s} = \{y \in S \mid s^*sy = y\} \text{ then } (sx)^*(sx) = x^*x$$

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Green's relations:

- $x \,\mathcal{L}\, y$ if $x^*x = y^*y$
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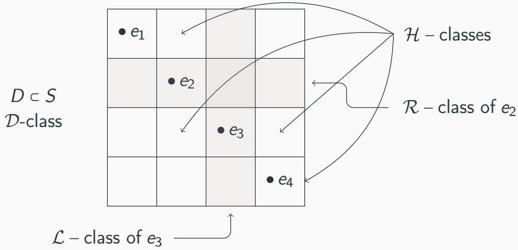

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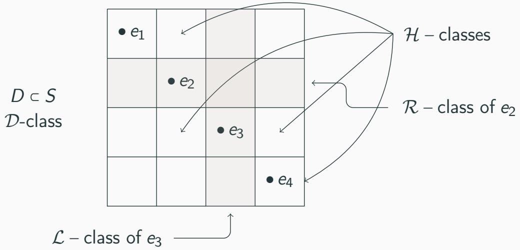

Remark - need of extended metrics

Good distances $d \colon S \times S \to [0, \infty]$ satisfy that

$$x^* x = y^* y \Leftrightarrow d(x, y) < \infty$$

Schützenberger graphs I: $\mathcal{L}$-classes

Definition (Schützenberger - 1959)

Let $S = \langle K \rangle$, where $K = K^*$. Given an $\mathcal{L}$-class $L \subset S$, let Λ_L be

- the graph whose vertices are the points of L and
- where $x, y \in L$ are joined by a k-labeled edge if $kx = y$.

Likewise, let $\Lambda_S = \sqcup_{e \in E(S)} \Lambda_{L_e}$.

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Proof/consequence:

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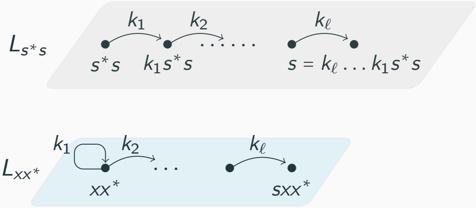

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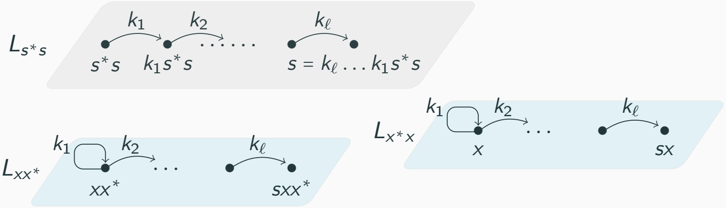

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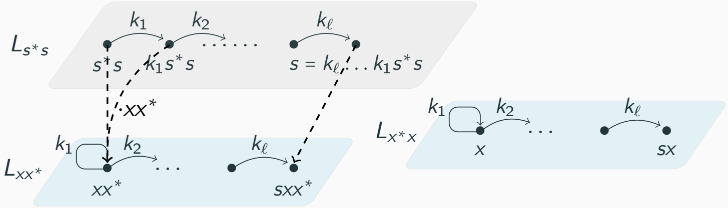

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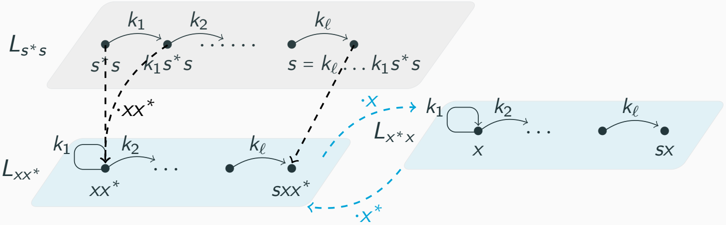

3. Day's amenability vs. nuclearity

Day's amenability as a coarse condition

Definition & Proposition (Day 1957 & Ara-Lledó-M. 2020)

S is *amenable* if there is $\mu: \mathcal{P}(S) \to [0,1]$ such that

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Theorem (Ara-Lledó-M. 2020)

S has a measure as in (b) $\Leftrightarrow$ there is $e \in E$ and $F_n \subset L_e$ such that

$$\frac{|s(F_n \cap D_{s^*s}) \cup F_n|}{|F_n|} \xrightarrow{n\to\infty} 1 \quad \text{for all } s \in S$$

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S has a measure as in (b) $\Leftrightarrow$ there is $e \in E$ and $F_n \subset L_e$ such that

$$\frac{|s(F_n \cap D_{s^*s}) \cup F_n|}{|F_n|} \xrightarrow{n\to\infty} 1 \quad \text{for all } s \in S$$

Consequence: (b) $\Leftrightarrow$ an $\mathcal{L}$-class has subsets with small boundary

Day's amenability vs. traces vs. nuclearity

Theorem (Ara-Lledó-M. 2020)

Let S be a unital inverse semigroup. TFAE:

(i) S has a probability measure as in (b).
(ii) $\mathcal{R}_S = C^*\left(\{fv_s\}_{s\in S, f\in\ell^\infty(S)}\right)$ has an (amenable) trace.
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WARNING: Day's amenability is **not** related with nuclearity!

Examples:

- $\mathbb{F}_2 \sqcup \{0\}$ is Day's amenable but not nuclear
- An example of Nica is nuclear but not Day's amenable

4. Property A vs. exactness

Schützenberger graphs and property A

Definition (Yu - 1999)

(X, d) has *property A* if for every $r, \varepsilon > 0$ there is

$$\xi\colon X \to \ell^1(X)_1^+ \text{ and } c > 0 \text{ such that } \operatorname{supp}(\xi_x) \subset B_c(x) \text{ and}$$

$$\|\xi_x - \xi_y\|_1 \leq \varepsilon \text{ for every } x, y \in X \text{ such that } d(x, y) \leq r.$$

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- Non-property A groups are hard to come by

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- Non-property A groups are hard to come by

Theorem (Ozawa - 2000)

Let G be a countable group. The following are equivalent:

(1) G has property A.
(2) $\mathcal{R}_G$ is nuclear.
(3) $C_r^*(G)$ is exact.

Property A, nuclearity and exactness for inverse semigroups

Theorem (Lledó, M. - 2021)

Let $S = \langle K \rangle$ be an inverse semigroup. *Some* times, TFAE:

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Proof: (i) $\Rightarrow$ (ii) given $\xi\colon S \to \ell^1(S)_1^+$ the diagram

$$\mathcal{R}_S \to \prod_{x \in S} M_{B_c(x)} \subset \ell^\infty(S) \otimes M_q \to \mathcal{R}_S$$

$$a \mapsto \left(p_{B_c(x)}\, a\, p_{B_c(x)}\right)_{x \in S} \rightsquigarrow (b_x)_{x \in S} \mapsto \sum_{x \in S} \xi_x^* b_x \xi_x$$

can be shown to be an approximation of $\mathrm{id}\colon \mathcal{R}_S \to \mathcal{R}_S$

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Question: some times? When is that?

5. Having finitely complex local structures

Finite labeling I: local structure

Note: Λ_S = vertices + edges, and edges represent (x, sx)

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Problem: $S = (\mathbb{N}, \min) \rightsquigarrow n \cdot m := \min\{n, m\}$

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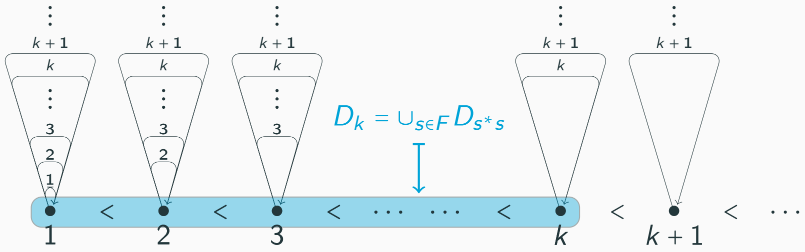

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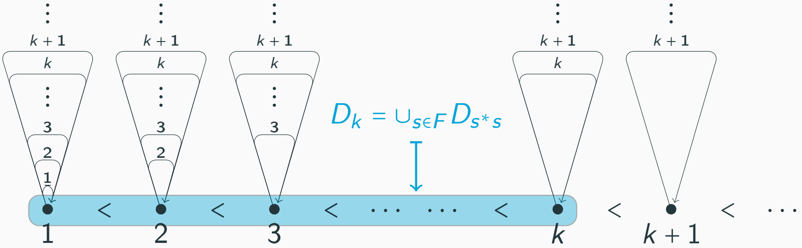

Key: for large $r \geq 0 \ \nexists \ F \Subset S$ labeling the paths in Λ_S of length r

Finite labeling II: a picture to *top* the explanation

Definition (Lledó, M. - 2021)

Let $S = \langle K \rangle$. We say (S, K) admits a *finite labeling* if

for any $r \geq 0$ there is $F \Subset S$ such that if $d(s^* s, s) \leq r$

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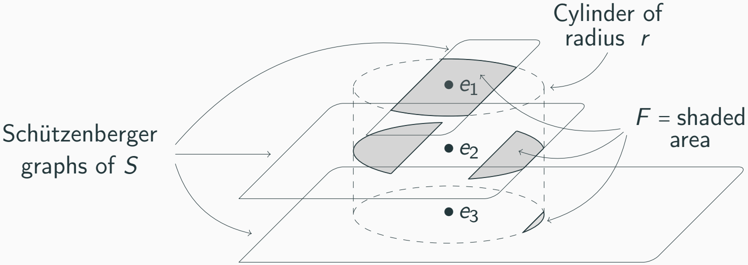

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Finite labeling III: importance

Theorem (Lledó, M. - 2021)

Suppose that (S, K) admits a finite labeling. Then:

$$\mathcal{R}_S \cong \ell^\infty(S) \rtimes_r S \cong C_u^*(\Lambda_S)$$

for a certain canonical action $S \curvearrowright \ell^\infty(S)$.

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Actually, even more is true:

Theorem (Lledó, M. - 2021)

$\mathcal{R}_S \cong C^*_u(\Lambda_S)$ if, and only if, (S, K) admits a FL.

Finite labelings characterizing the uniform Roe algebra

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Recall $C_u^*(\Lambda_S) = C^*\left(\left\{t \in \mathcal{B}\left(\ell^2(S)\right) \text{ of finite propagation }\right\}\right)$

Proof:

- If t has finite propagation $+$ (S, K) admits a FL
  then $\rightsquigarrow$ $t = \sum_{s \in F} f_s v_s$, where $f_s \in \ell^\infty(S)$
  i.e., we can label the pairs (x, y) with $\langle \delta_y, t\delta_x \rangle \neq 0$ as pairs (x, sx) with $s \in F \Subset S$. Thus $\mathcal{R}_S \supset C_u^*(\Lambda_S)$
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Thank you for your attention!

Questions?