C* vs coarse properties of inverse semigroups

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Analysis seminar - University of Glasgow

Based on joint work with Pere Ara and Fernando Lledó

Outline

(1) Inverse semigroups

(2) Schützenberger graphs and right invariance

(3) Day's amenability vs. nuclearity

(4) Property A vs. exactness

(5) *Having finite local structure*

1. Inverse semigroups

Inverse semigroups and Wagner-Preston

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- $E = \left\{ e \in S \mid e^2 = e \right\} = \{ s^*s \mid s \in S \}$ is commutative
- $\underline{D_{s^*s} := \{ x \in S \mid x = s^*sx \}} = s^*s \cdot S$ is the *domain of s*
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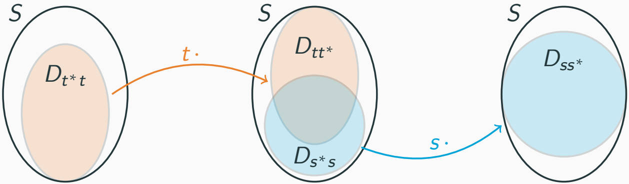

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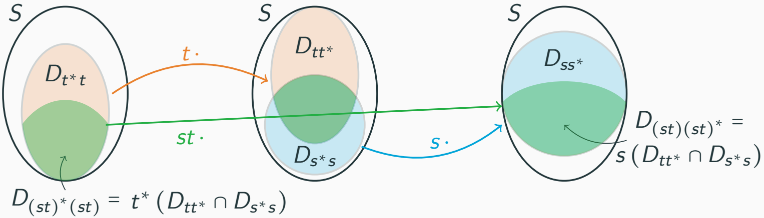

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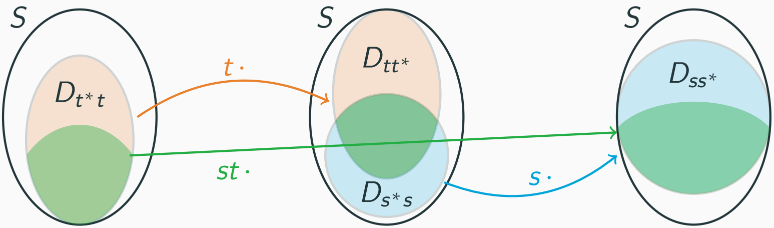

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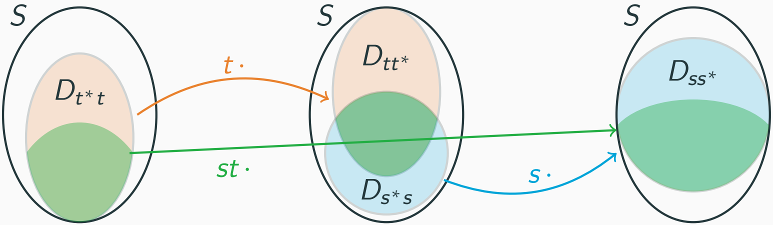

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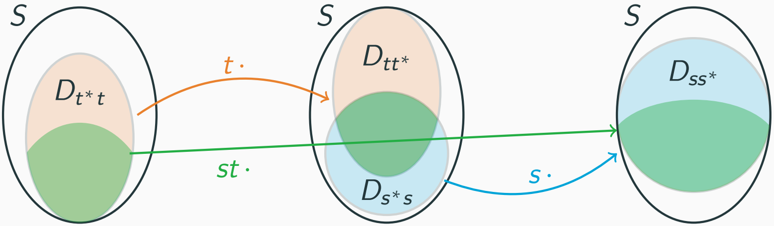

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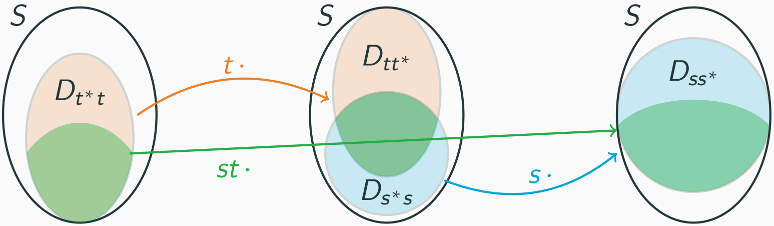

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<u>**Reduced C*-algebra:**</u> $C_r^*(S) := C^*\left(\{v_s\}_{s\in S}\right) \subset \mathcal{B}\left(\ell^2(S)\right)$

<u>**Uniform Roe algebra:**</u> $\mathcal{R}_S := C^*\left(\{fv_s\}_{s\in S, f\in\ell^\infty(S)}\right) \subset \mathcal{B}\left(\ell^2(S)\right)$

Group coarse geometry

Recall: Cayley graph construction $\rightsquigarrow G = \langle\, g_1^{\pm 1}, \ldots, g_n^{\pm 1} \,|\, \text{relations} \,\rangle$:

- Graph $\rightsquigarrow \operatorname{Cay}(G, \{g_1, \ldots, g_n\}) := (V, E)$,
- Vertices $\rightsquigarrow V := G$
- Edges $\rightsquigarrow E := \big\{\big(x, g_i^{\pm 1} x\big) \mid x \in G \text{ and } i = 1, \ldots, n\big\}$.

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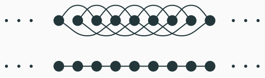

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Proposition (classical)

The large scale geometry of the Cayley graph of G does **not** depend on the generators

Goal: coarse geometry of inverse semigroups and its relation with C*-properties of $C_r^*(S)$ and $\mathcal{R}_S$

2. Schützenberger graphs and right invariance

Infinite distances, and why they are necessary

Remark: we need to consider *extended* metric spaces:

$$\text{if } x \in D_{s^*s} = \{y \in S \mid s^* s y = y\} \text{ then } (sx)^* (sx) = x^* x$$

and hence $x \mathcal{L} sx$ (the converse also holds)

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Green's relations:

- $x \mathcal{L} y$ if $x^*x = y^*y$
- $x \mathcal{R} y$ if $xx^* = yy^*$
- $\mathcal{H} = \mathcal{L}$ and $\mathcal{R}$
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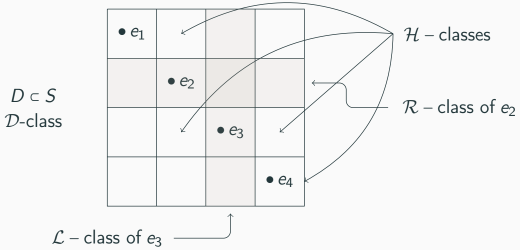

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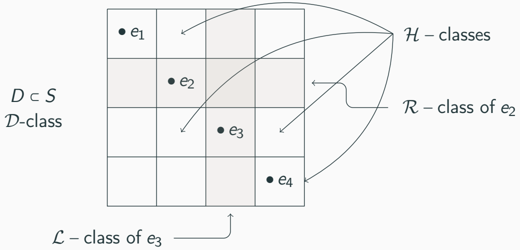

Remark - need of extended metrics

Good distances $d \colon S \times S \to [0, \infty]$ satisfy that

$$x^*x = y^*y \Leftrightarrow d(x, y) < \infty$$

Schützenberger graphs I: $\mathcal{L}$-classes

Definition (Schützenberger - 1959)

Let $S = \langle K \rangle$, where $K = K^*$. Given an $\mathcal{L}$-class $L \subset S$, let Λ_L be

- the graph whose vertices are the points of L and
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Likewise, let $\Lambda_S = \sqcup_{e \in E(S)} \Lambda_{L_e}$.

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Remark: not all graphs are group Cayley graphs. However:

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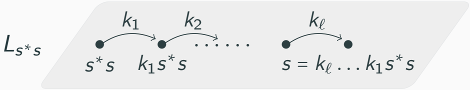

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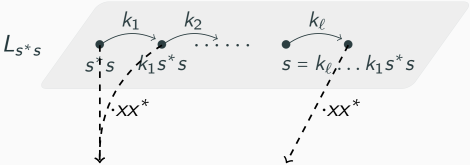

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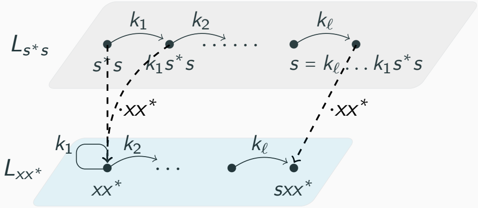

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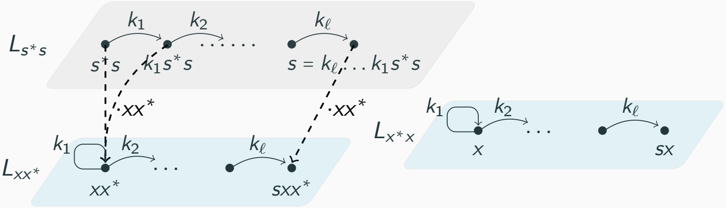

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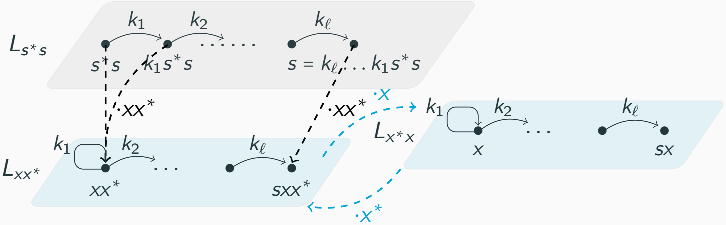

3. Day's amenability vs. nuclearity

Day's amenability as a coarse condition

Definition & Proposition (Day 1957 & Ara-Lledó-M. 2020)

S is *amenable* if there is $\mu \colon \mathcal{P}(S) \to [0,1]$ such that

(a) $\mu(A) = \mu(A \cap D_{s^*s})$ for every $s \in S$ and $A \subset S$
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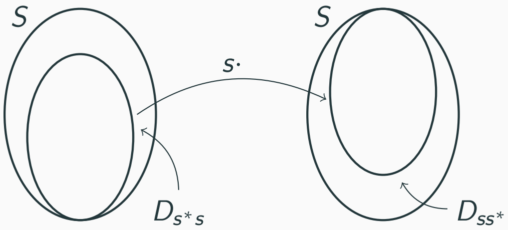

Day's amenability as a coarse condition

Definition & Proposition (Day 1957 & Ara-Lledó-M. 2020)

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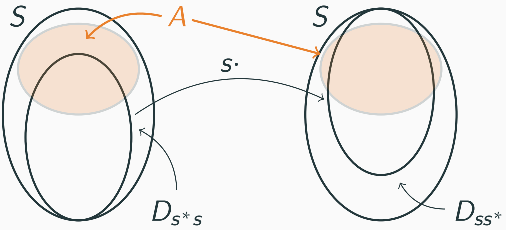

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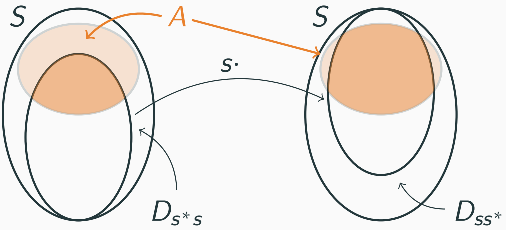

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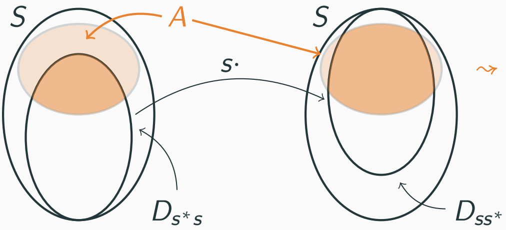

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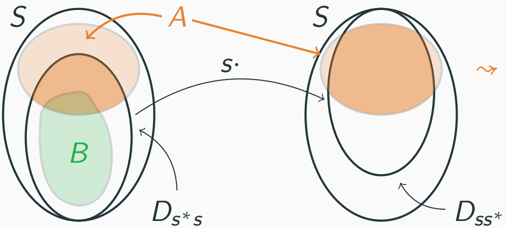

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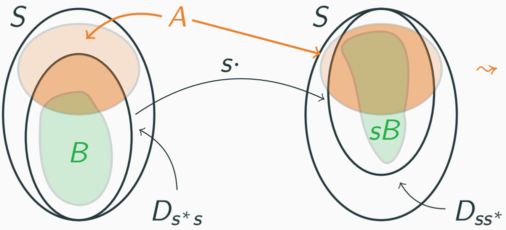

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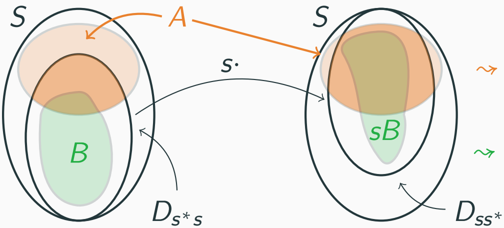

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- $\mathbb{F}_2 \sqcup \{1\}$ is domain-measurable but non-amenable

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Theorem (Ara-Lledó-M. 2020)

S is domain-measurable $\Leftrightarrow$ there is $e \in E$ and $F_n \subset L_e$ such that

$$\frac{|s(F_n \cap D_{s^*s}) \cup F_n|}{|F_n|} \xrightarrow{n\to\infty} 1 \text{ for all } s \in S$$

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Consequence: domain-measurability $\Leftrightarrow$

an $\mathcal{L}$-class has subsets with small boundary, i.e., Følner sets

Day's amenability vs. traces

Theorem (Ara-Lledó-M. 2020)

Let S be a unital inverse semigroup. TFAE:

(i) S is domain-measurable.
(ii) $\mathcal{R}_S = C^*\left(\{fv_s\}_{s\in S, f\in\ell^\infty(S)}\right)$ has an (amenable) trace.
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Proof: (i) $\Leftrightarrow$ (iii) by the discussion before

For (i) $\Rightarrow$ (ii), given $\mu: \mathcal{P}(S) \to [0,1]$ the following is a trace

$$\tau_\mu: \mathcal{R}_S \to \ell^\infty(S) \xrightarrow{m_\mu} \mathbb{C},$$
$$a \mapsto \sum_{x\in S} \langle a\delta_x, \delta_x\rangle, \text{ and } m_\mu(p_A) = \mu(A).$$

Lastly, (ii) $\Rightarrow$ (iii) involves a suitably adapted Namioka's trick...

Day's amenability vs. nuclearity

Remark: condition (a) can also be plugged above by saying:

- The trace τ_μ satisfies $\tau_\mu(v_e) = 1$ for every $e \in E$
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WARNING: Day's amenability is **not** related with nuclearity!

Examples:

- $\mathbb{F}_2 \sqcup \{0\}$ is Day's amenable but not nuclear
- An example of Nica is nuclear but not Day's amenable

4. Property A vs. exactness

Schützenberger graphs and property A

Definition (Yu - 1999)

(X, d) has *property A* if for every $r, \varepsilon > 0$ there is

$$\xi \colon X \to \ell^1(X)_1^+ \text{ and } c > 0 \text{ such that } \operatorname{supp}(\xi_x) \subset B_c(x) \text{ and}$$

$$\|\xi_x - \xi_y\|_1 \leq \varepsilon \text{ for every } x, y \in X \text{ such that } d(x, y) \leq r$$

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- Property A generalizes amenability <u>for groups</u> (**not** in general)
- Non-property A groups are hard to come by

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Remarks:

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- Non-property A groups are hard to come by

Theorem (Ozawa - 2000)

Let G be a countable group. The following are equivalent:

(1) G has property A.
(2) $\mathcal{R}_G$ is nuclear.
(3) $C^*_r(G)$ is exact.

Property A, nuclearity and exactness for inverse semigroups

Theorem (Lledó, M. - 2021)

Let $S = \langle K \rangle$ be an inverse semigroup. *Some* times, TFAE:

(i) Λ_S has property A.
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Proof: (i) $\Rightarrow$ (ii) given $\xi\colon S \to \ell^1(S)_1^+$ the diagram

$$\mathcal{R}_S \to \prod_{x\in S} M_{B_c(x)} \subset \ell^\infty(S) \otimes M_q \to \mathcal{R}_S$$

$$a \mapsto \left(p_{B_c(x)}\, a\, p_{B_c(x)}\right)_{x\in S} \rightsquigarrow (b_x)_{x\in S} \mapsto \sum_{x\in S} \xi_x^* b_x \xi_x$$

can be shown to be an approximation of $\mathrm{id}\colon \mathcal{R}_S \to \mathcal{R}_S$

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Question: some times? When is that?

5. Having finitely complex local structures

Finite labeling I: local structure

Note: Λ_S = vertices + edges, and edges represent (x, sx)

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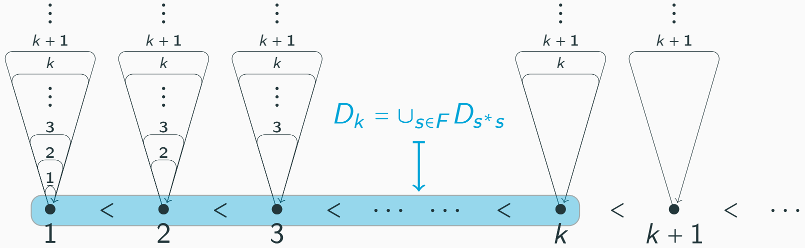

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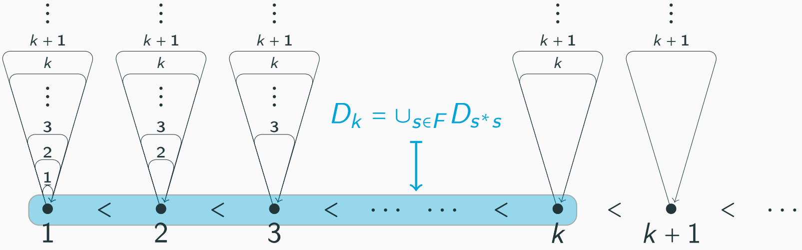

Key: for large $r \geq 0 \nexists F \Subset S$ labeling the paths in Λ_S of length r

Finite labeling II: a picture to *top* the explanation

Definition (Lledó, M. - 2021)

Let $S = \langle K \rangle$. We say (S, K) admits a *finite labeling* if

for any $r \geq 0$ there is $F \Subset S$ such that if $d(s^*s, s) \leq r$

then there is $m \in F$ such that $ms^*s = s$.

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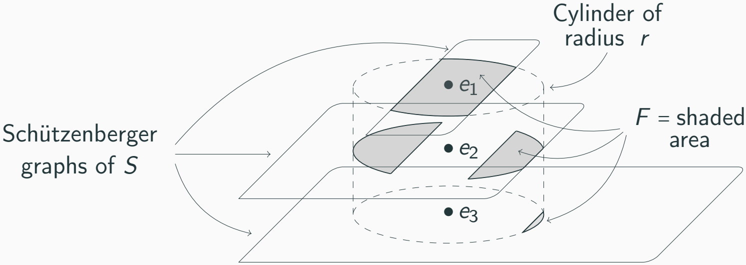

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Finite labeling III: importance

Theorem (Lledó, M. - 2021)

Suppose that (S, K) admits a finite labeling. Then:

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Actually, even more is true:

Theorem (Lledó, M. - 2021)

$$\mathcal{R}_S \cong C^*_u(\Lambda_S) \iff (S, K) \text{ admits a FL.}$$

Finite labelings characterizing the uniform Roe algebra

Theorem (Lledó, M. - 2021)

$$\mathcal{R}_S \cong C_u^*(\Lambda_S) \iff (S, K) \text{ admits a FL.}$$

Recall $C_u^*(\Lambda_S) = C^*\left(\left\{t \in \mathcal{B}\left(\ell^2(S)\right) \text{ of finite propagation }\right\}\right)$

Proof:

- If t has finite propagation + (S, K) admits a FL
  then $\rightsquigarrow$ $t = \sum_{s \in F} f_s v_s$, where $f_s \in \ell^\infty(S)$
  i.e., we can label the pairs (x, y) with $\langle \delta_y, t\delta_x \rangle \neq 0$ as pairs (x, sx) with $s \in F \Subset S$. Thus $\mathcal{R}_S \supset C_u^*(\Lambda_S)$
- If not FL $\rightsquigarrow$ construct t of propagation 1 with $t \notin \mathcal{R}_S$

Conclusions and future work

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- Inverse semigroups can be seen geometrically

  Coarse geometry $\leftrightsquigarrow$ C*-properties
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Thank you for your attention!

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