Exactness and geometric properties of inverse semigroups

Diego Martínez – WWU Münster

March 17th, 2022

Non-commutativity in the North – Göteborgs Universitet

Based on joint work with Chyuan and Szakács

Outline

(1) Inverse semigroups

(2) Proper and right sub-invariant metrics

(3) Exactness vs. Yu's property A

(4) Asymptotic dimension 0 vs. local AF

1. Inverse semigroups

Inverse semigroups and Wagner-Preston

Definition

S is an *inverse semigroup* if for every $s \in S$ there is a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$

Definition

S is an *inverse semigroup* if for every $s \in S$ there is a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$

Remarks:

- *Bicyclic monoid:* $\mathcal{B} = \langle a, a^* \mid a^*a = 1 \rangle = \left\{ a^i a^{*j} \mid i, j \geq 0 \right\}$

Inverse semigroups and Wagner-Preston

Definition

S is an <u>*inverse semigroup*</u> if for every $s \in S$ there is a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$

Remarks:

- *Bicyclic monoid:* $\mathcal{B} = \langle a, a^* \mid a^*a = 1 \rangle = \left\{ a^i a^{*j} \mid i, j \geq 0 \right\}$
- $E = \left\{ e \in S \mid e^2 = e \right\} = \{ s^*s \mid s \in S \}$ is commutative
- $D_{s^*s} = s^*s \cdot S$ is the <u>*domain of s*</u>
- $s \colon D_{s^*s} \to D_{ss^*}$, where $x \mapsto sx$ is a bijection

Inverse semigroups and Wagner-Preston

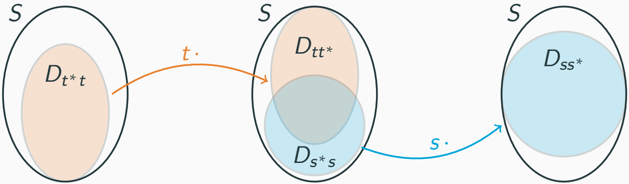
Definition

S is an <u>*inverse semigroup*</u> if for every $s \in S$ there is a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$

Remarks:

- *Bicyclic monoid:* $\mathcal{B} = \langle a, a^* \mid a^*a = 1 \rangle = \left\{ a^i a^{*j} \mid i, j \geq 0 \right\}$
- $E = \left\{ e \in S \mid e^2 = e \right\} = \left\{ s^*s \mid s \in S \right\}$ is commutative
- $D_{s^*s} = s^*s \cdot S$ is the <u>*domain of s*</u>
- $s \colon D_{s^*s} \to D_{ss^*}$, where $x \mapsto sx$ is a bijection

Induces the ***Wagner-Preston*** representation $\nu \colon S \to \mathcal{I}(S)$:



Inverse semigroups and Wagner-Preston

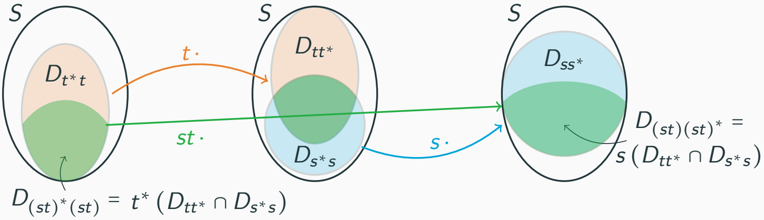
Definition

S is an *inverse semigroup* if for every $s \in S$ there is a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$

Remarks:

- *Bicyclic monoid:* $\mathcal{B} = \langle a, a^* \mid a^*a = 1 \rangle = \left\{ a^i a^{*j} \mid i, j \geq 0 \right\}$
- $E = \left\{ e \in S \mid e^2 = e \right\} = \left\{ s^*s \mid s \in S \right\}$ is commutative
- $D_{s^*s} = s^*s \cdot S$ is the *domain of s*
- $s \colon D_{s^*s} \to D_{ss^*}$, where $x \mapsto sx$ is a bijection

Induces the ***Wagner-Preston*** representation $v \colon S \to \mathcal{I}(S)$:



Remark: the Wagner-Preston representation encapsulates the idea behind how we see these inverse semigroups:

Heuristic

Usual use of inverse semigroups: $S \subset \mathrm{Bis}(G)$

$$\mathrm{Bis}(G) := \{u \subset G \mid \text{open and } r\colon u \to r(u) \text{ homeomorphism}\}$$

$s \in S$ is a **label** for an **open bunch of arrows** in a groupoid

Open bisections and partial order

Remark: the Wagner-Preston representation encapsulates the idea behind how we see these inverse semigroups:

Heuristic

Usual use of inverse semigroups: $S \subset \mathrm{Bis}(G)$

$\mathrm{Bis}(G) := \{u \subset G \mid \text{open and } r\colon u \to r(u) \text{ homeomorphism}\}$

$s \in S$ is a **label** for an **open bunch of arrows** in a groupoid

Partial order: $s \geq t \Leftrightarrow$ there is some $e \in E$ with $se = t$,

$\Leftrightarrow$ t is a *restriction* of $s \Leftrightarrow st^*t = t$

Open bisections and partial order

Remark: the Wagner-Preston representation encapsulates the idea behind how we see these inverse semigroups:

Heuristic

Usual use of inverse semigroups: $S \subset \mathrm{Bis}(G)$

$\mathrm{Bis}(G) := \{u \subset G \mid \text{open and } r\colon u \to r(u) \text{ homeomorphism}\}$

$s \in S$ is a **label** for an **open bunch of arrows** in a groupoid

Partial order: $s \geq t \Leftrightarrow$ there is some $e \in E$ with $se = t$,

$\Leftrightarrow t$ is a *restriction* of $s \Leftrightarrow st^*t = t$

Left regular representation: $v\colon S \to \mathcal{B}\left(\ell^2(S)\right)$, where

$$v_s\delta_x = \begin{cases} \delta_{sx} & \text{if } x \in s^*s \cdot S \\ 0 & \text{otherwise} \end{cases}$$

Open bisections and partial order

Remark: the Wagner-Preston representation encapsulates the idea behind how we see these inverse semigroups:

Heuristic

Usual use of inverse semigroups: $S \subset \mathrm{Bis}(G)$

$\mathrm{Bis}(G) := \{u \subset G \mid \text{open and } r\colon u \to r(u) \text{ homeomorphism}\}$

$s \in S$ is a **label** for an **open bunch of arrows** in a groupoid

Partial order: $s \geq t \Leftrightarrow$ there is some $e \in E$ with $se = t$,

$\Leftrightarrow t$ is a *restriction* of $s \Leftrightarrow st^* t = t$

Left regular representation: $v\colon S \to \mathcal{B}\left(\ell^2(S)\right)$, where

$$v_s \delta_x = \begin{cases} \delta_{sx} & \text{if } x \in s^* s \cdot S \\ 0 & \text{otherwise} \end{cases}$$

Reduced C*-algebra: $C_r^*(S) := C^*\left(\{v_s\}_{s \in S}\right) \subset \mathcal{B}\left(\ell^2(S)\right)$

Group coarse geometry

Recall: Cayley graph construction $\rightsquigarrow G = \langle\, g_1^{\pm 1}, \ldots, g_n^{\pm 1} \,|\, \text{relations} \,\rangle$:

- Graph $\rightsquigarrow \mathrm{Cay}\,(G, \{g_1, \ldots, g_n\}) := (V, E)$,
- Vertices $\rightsquigarrow V := G$
- Edges $\rightsquigarrow E := \left\{\left(x, g_i^{\pm 1} x\right) \mid x \in G \text{ and } i = 1, \ldots, n\right\}$.

Group coarse geometry

Recall: Cayley graph construction $\rightsquigarrow G = \langle\, g_1^{\pm 1}, \ldots, g_n^{\pm 1} \,|\, \text{relations} \,\rangle$:

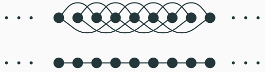
- Graph $\rightsquigarrow \mathrm{Cay}\,(\,G, \{g_1, \ldots, g_n\}) := (\,V, E\,)$,
- Vertices $\rightsquigarrow V := G$
- Edges $\rightsquigarrow E := \left\{\left(x, g_i^{\pm 1} x\right) \mid x \in G \text{ and } i = 1, \ldots, n\right\}$.

$\cdots$  $\cdots$ $\mathbb{Z} = \langle \pm 2, \pm 3 \rangle$

$\cdots$  $\cdots$ $\mathbb{Z} = \langle \pm 1 \rangle$

Recall: Cayley graph construction $\rightsquigarrow G = \langle\, g_1^{\pm 1}, \ldots, g_n^{\pm 1} \,|\, \text{relations} \,\rangle$:

- Graph $\rightsquigarrow \operatorname{Cay}(G, \{g_1, \ldots, g_n\}) := (V, E)$,
- Vertices $\rightsquigarrow V := G$
- Edges $\rightsquigarrow E := \left\{ \left(x, g_i^{\pm 1} x\right) \mid x \in G \text{ and } i = 1, \ldots, n \right\}$.



$\mathbb{Z} = \langle \pm 2, \pm 3 \rangle$

$\mathbb{Z} = \langle \pm 1 \rangle$

Proposition (classical)

The large scale geometry of the Cayley graph of G
does **not** depend on the generators

Goal: coarse geometry of inverse semigroups
and its relation with C*-properties of $C_r^*(S)$

2. Proper and right sub-invariant metrics

Infinite distances, and why they are necessary

Remark: we need to consider *extended* metric spaces:

$$S := G \sqcup \{0\} \rightsquigarrow s \cdot 0 = 0 \cdot s = 0 = s^* \cdot 0 = 0 \cdot s^*$$

and, hence, there is a **directed** edge $s \to 0 \quad \rightsquigarrow \quad d(s, 0) = \infty$

Infinite distances, and why they are necessary

Remark: we need to consider *extended* metric spaces:

$$S := G \sqcup \{0\} \rightsquigarrow s \cdot 0 = 0 \cdot s = 0 = s^* \cdot 0 = 0 \cdot s^*$$

and, hence, there is a **directed** edge $s \to 0 \quad \rightsquigarrow \quad d(s, 0) = \infty$

Definition (Chung, M. and Szakács - 22)

Let $d\colon S \times S \to [0, \infty]$. We say d *respects the components of S* if

$$d(s, t) < \infty \quad \Leftrightarrow \quad s^* s = t^* t$$

Infinite distances, and why they are necessary

Remark: we need to consider *extended* metric spaces:

$$S := G \sqcup \{0\} \rightsquigarrow s \cdot 0 = 0 \cdot s = 0 = s^* \cdot 0 = 0 \cdot s^*$$

and, hence, there is a **directed** edge $s \to 0 \quad \rightsquigarrow \quad d(s,0) = \infty$

Definition (Chung, M. and Szakács - 22)

Let $d\colon S \times S \to [0,\infty]$. We say d *respects the components of S* if

$$d\,(s,t) < \infty \quad \Leftrightarrow \quad s^* s = t^* t$$

Remark:

- Automatic for groups
- $s^* s = 0^* 0 = 0 \;\Rightarrow\; s = s s^* s = s0 = 0,$

  and therefore $\{0\} \subset S$ forms a component!
- $(S,d) = \sqcup_{e \in E} (L_e, d|_{L_e})$ are the connected components
- This allows for uncountable S

Infinite distances, and why they are necessary

Remark: we need to consider *extended* metric spaces:

$$S := G \sqcup \{0\} \rightsquigarrow s \cdot 0 = 0 \cdot s = 0 = s^* \cdot 0 = 0 \cdot s^*$$

and, hence, there is a **directed** edge $s \to 0 \rightsquigarrow d(s,0) = \infty$

Definition (Chung, M. and Szakács - 22)

Let $d\colon S \times S \to [0,\infty]$. We say d *respects the components of S* if

$$d(s,t) < \infty \quad \Leftrightarrow \quad s^*s = t^*t$$

Remark:

- Automatic for groups
- $s^*s = 0^*0 = 0 \;\Rightarrow\; s = ss^*s = s0 = 0$,

  and therefore $\{0\} \subset S$ forms a component!
- $(S,d) = \sqcup_{e\in E}(L_e, d|_{L_e})$ are the connected components
- This allows for uncountable S

Standing assumption: d respects the components of S

Proper and right sub-invariant metrics

Definition (Chung, M. and Szakács - 22)

Let $d\colon S \times S \to [0, \infty]$ be a metric. We say d is:

- ***right sub-invariant*** if $d(sr, tr) \leq d(s, t)$ for all $s, t, r \in S$
- ***proper*** if for all $r \geq 0$ there is a finite $F \Subset S$ such that $t \in Fs$ for all $s, t \in S$ such that $d(s, t) \leq r$.

Definition (Chung, M. and Szakács - 22)

Let $d\colon S \times S \to [0, \infty]$ be a metric. We say d is:

- ***right sub-invariant*** if $d(sr, tr) \leq d(s, t)$ for all $s, t, r \in S$
- ***proper*** if for all $r \geq 0$ there is a finite $F \Subset S$ such that $t \in Fs$ for all $s, t \in S$ such that $d(s, t) \leq r$.

Remarks:

- Generalizes properness and right invariance for groups

Proper and right sub-invariant metrics

Definition (Chung, M. and Szakács - 22)

Let $d\colon S \times S \to [0, \infty]$ be a metric. We say d is:

- ***right sub-invariant*** if $d(sr, tr) \leq d(s, t)$ for all $s, t, r \in S$
- ***proper*** if for all $r \geq 0$ there is a finite $F \Subset S$ such that $t \in Fs$ for all $s, t \in S$ such that $d(s, t) \leq r$.

Remarks:

- Generalizes properness and right invariance for groups

  For instance, if $S = \sqcup_{e \in E} G_e$, then $(S, d) = \sqcup_{e \in E} (G_e, d|_{G_e})$
- $S = \langle s_1, \ldots, s_k \mid \text{relations} \rangle \rightsquigarrow d$ is the path metric in $\{\Lambda_e\}_{e \in E}$

Definition (Chung, M. and Szakács - 22)

Let $d\colon S \times S \to [0, \infty]$ be a metric. We say d is:

- ***right sub-invariant*** if $d(sr, tr) \leq d(s,t)$ for all $s, t, r \in S$
- ***proper*** if for all $r \geq 0$ there is a finite $F \Subset S$ such that $t \in Fs$ for all $s, t \in S$ such that $d(s,t) \leq r$.

Remarks:

- Generalizes properness and right invariance for groups
  For instance, if $S = \sqcup_{e \in E} G_e$, then $(S, d) = \sqcup_{e \in E} (G_e, d|_{G_e})$
- $S = \langle s_1, \ldots, s_k \mid \text{relations} \rangle \rightsquigarrow d$ is the path metric in $\{\Lambda_e\}_{e \in E}$
- If d is proper, then (S, d) has bounded geometry
  However, the converse is false!

Existence and uniqueness of these metrics

Theorem (Chung, M. and Szakács - 22)

Every countable inverse semigroup has a proper and right sub-invariant metric. Moreover, such a metric is **unique** up to bijective coarse equivalence.

Existence and uniqueness of these metrics

Theorem (Chung, M. and Szakács - 22)

Every countable inverse semigroup has a proper and right sub-invariant metric. Moreover, such a metric is **unique** up to bijective coarse equivalence.

Remark: works for some non-countable semigroups...
as long as $S = \langle F \cup E \rangle$, where F is countable

For instance: an action $G \curvearrowright$ Cantor, where G is a discrete group, induces $S = \mathrm{Bis}\,(G \curvearrowright \text{Cantor})$ as above

Existence and uniqueness of these metrics

Theorem (Chung, M. and Szakács - 22)

Every countable inverse semigroup has a proper and right sub-invariant metric. Moreover, such a metric is **unique** up to bijective coarse equivalence.

Remark: works for some non-countable semigroups...
as long as $S = \langle F \cup E \rangle$, where F is countable

For instance: an action $G \curvearrowright$ Cantor, where G is a discrete group, induces $S = \mathrm{Bis}\,(G \curvearrowright \mathrm{Cantor})$ as above

Question: what sort of metric spaces (S, d) can we get?

Existence and uniqueness of these metrics

Theorem (Chung, M. and Szakács - 22)

Every countable inverse semigroup has a proper and right sub-invariant metric. Moreover, such a metric is **unique** up to bijective coarse equivalence.

Remark: works for some non-countable semigroups... as long as $S = \langle F \cup E \rangle$, where F is countable

For instance: an action $G \curvearrowright$ Cantor, where G is a discrete group, induces $S = \mathrm{Bis}\,(G \curvearrowright \text{Cantor})$ as above

Question: what sort of metric spaces (S, d) can we get?

Theorem (Chung, M. and Szakács - 22)

Any (X, d) of bounded geometry is a component of some inverse semigroup (that depends on X)

3. Exactness vs. Yu's property A

Metric spaces and property A

Definition (Yu - 1999)

(X, d) has *property A* if for every $r, \varepsilon > 0$ there is

$$\xi \colon X \to \ell^1(X)^+_1 \text{ and } c > 0 \text{ such that } \operatorname{supp}(\xi_x) \subset B_c(x) \text{ and}$$

$$\|\xi_x - \xi_y\|_1 \leq \varepsilon \text{ for every } x, y \in X \text{ such that } d(x, y) \leq r$$

Metric spaces and property A

Definition (Yu - 1999)

(X,d) has *property A* if for every $r, \varepsilon > 0$ there is

$$\xi \colon X \to \ell^1(X)_1^+ \text{ and } c > 0 \text{ such that } \operatorname{supp}(\xi_x) \subset B_c(x) \text{ and}$$

$$\|\xi_x - \xi_y\|_1 \leq \varepsilon \text{ for every } x, y \in X \text{ such that } d(x,y) \leq r$$

Remarks:

- Property A generalizes amenability for groups (**not** in general)
- Non-property A groups are hard to come by

Metric spaces and property A

Definition (Yu - 1999)

(X,d) has *property A* if for every $r,\varepsilon>0$ there is
$\xi\colon X\to\ell^1(X)_1^+$ and $c>0$ such that $\operatorname{supp}(\xi_x)\subset B_c(x)$ and
$\|\xi_x-\xi_y\|_1\le\varepsilon$ for every $x,y\in X$ such that $d(x,y)\le r$

Remarks:

- Property A generalizes amenability for groups (**not** in general)
- Non-property A groups are hard to come by

Theorem (Ozawa - 2000)

Let G be a countable group. TFAE:

(1) (G,d) has property A, where d is proper and r.inv.
(2) $\ell^\infty(G)\rtimes_r G$ is nuclear.
(3) $C_r^*(G)$ is exact.

Property A, nuclearity and exactness for inverse semigroups

Theorem (Lledó, M. - 2021, and Alcides, M. - 2022)

Let S be a countable inverse semigroup. TFAE:

(i) (S, d) has property A, where d is proper and r.inv.
(ii) $\ell^\infty(S) \rtimes_r S$ is nuclear.
(iii) $C_r^*(S)$ is exact.

Theorem (Lledó, M. - 2021, and Alcides, M. - 2022)

Let S be a countable inverse semigroup. TFAE:

(i) (S, d) has property A, where d is proper and r.inv.
(ii) $\ell^\infty(S) \rtimes_r S$ is nuclear.
(iii) $C_r^*(S)$ is exact.

Proof: (i) $\Rightarrow$ (ii) given $\xi\colon S \to \ell^1(S)_1^+$ the diagram

$$\mathcal{R}_S \to \prod_{x\in S} M_{B_c(x)} \subset \ell^\infty(S) \otimes M_q \to \mathcal{R}_S$$

$$a \mapsto \left(p_{B_c(x)}\, a\, p_{B_c(x)}\right)_{x\in S} \rightsquigarrow (b_x)_{x\in S} \mapsto \sum_{x\in S} \xi_x^* b_x \xi_x$$

can be shown to be an approximation of id: $\mathcal{R}_S \to \mathcal{R}_S$

(ii) $\Rightarrow$ (iii) is clear, while

(iii) $\Rightarrow$ (i) is based on $\ell^\infty(S) \rtimes_r S \cong C_u^*(S, d)$

4. Asymptotic dimension 0 vs. local AF algebras

Semigroups of asymptotic dimension 0

Recall: $\operatorname{asdym}(X, d) = 0$ is an analog for being a Cantor set

Definition

$\operatorname{asdim}(X, d) = 0$ if for every $r \geq 0, X$ has a partition $\mathcal{U}$ such that

$$\inf_{U \neq V \in \mathcal{U}} d(U, V) \geq r \quad \text{and} \quad \sup_{U \in \mathcal{U}} \operatorname{diam}(U) < \infty$$

Semigroups of asymptotic dimension 0

Recall: $\operatorname{asdym}(X, d) = 0$ is an analog for being a Cantor set

Definition

$\operatorname{asdim}(X, d) = 0$ if for every $r \geq 0, X$ has a partition $\mathcal{U}$ such that

$$\inf_{U \neq V \in \mathcal{U}} d(U, V) \geq r \quad \text{and} \quad \sup_{U \in \mathcal{U}} \operatorname{diam}(U) < \infty$$

Question: when does S have asymptotic dimension 0?

Recall: $\text{asdym}(X, d) = 0$ is an analog for being a Cantor set

Definition

$\text{asdim}\,(X, d) = 0$ if for every $r \geq 0, X$ has a partition $\mathcal{U}$ such that

$$\inf_{U \neq V \in \mathcal{U}} d\,(U, V) \geq r \quad \text{and} \quad \sup_{U \in \mathcal{U}} \text{diam}\,(U) < \infty$$

Question: when does S have asymptotic dimension 0?

Answers:

- If S is finite then $\text{asdim}(S) = 0$

Semigroups of asymptotic dimension 0

Recall: $\mathrm{asdym}(X, d) = 0$ is an analog for being a Cantor set

Definition

$\mathrm{asdim}\,(X, d) = 0$ if for every $r \geq 0, X$ has a partition $\mathcal{U}$ such that

$$\inf_{U \neq V \in \mathcal{U}} d\,(U, V) \geq r \quad \text{and} \quad \sup_{U \in \mathcal{U}} \mathrm{diam}\,(U) < \infty$$

Question: when does S have asymptotic dimension 0?

Answers:

- If S is finite then $\mathrm{asdim}(S) = 0$
- If S is fin. gen., then S finite iff $\mathrm{asdim}(S) = 0$

Semigroups of asymptotic dimension 0

Recall: $\operatorname{asdym}(X, d) = 0$ is an analog for being a Cantor set

Definition

asdim $(X, d) = 0$ if for every $r \geq 0, X$ has a partition $\mathcal{U}$ such that

$$\inf_{U \neq V \in \mathcal{U}} d\,(U, V) \geq r \quad \text{and} \quad \sup_{U \in \mathcal{U}} \operatorname{diam}(U) < \infty$$

Question: when does S have asymptotic dimension 0?

Answers:

- If S is finite then $\operatorname{asdim}(S) = 0$
- If S is fin. gen., then S finite iff $\operatorname{asdim}(S) = 0$
- If we add new generators $S = \langle \{t_1, \ldots, t_n\} \cup \{s_1, \ldots, s_m\} \rangle$ then
$$\sup_{j=1,\ldots,n} d\left(t_j^* t_j, t_j\right) < \inf_{i=1,\ldots,m} d\left(s_i^* s_i, s_i\right),$$
and that doesn't increase the asymptotic dimension

Semigroups of asymptotic dimension 0

Recall: $\operatorname{asdym}(X, d) = 0$ is an analog for being a Cantor set

Definition

$\operatorname{asdim}(X, d) = 0$ if for every $r \geq 0, X$ has a partition $\mathcal{U}$ such that

$$\inf_{U \neq V \in \mathcal{U}} d(U, V) \geq r \quad \text{and} \quad \sup_{U \in \mathcal{U}} \operatorname{diam}(U) < \infty$$

Question: when does S have asymptotic dimension 0?

Answers:

- If S is finite then $\operatorname{asdim}(S) = 0$
- If S is fin. gen., then S finite iff $\operatorname{asdim}(S) = 0$
- If we add new generators $S = \langle \{t_1, \ldots, t_n\} \cup \{s_1, \ldots, s_m\} \rangle$ then
$$\sup_{j=1,\ldots,n} d\left(t_j^* t_j, t_j\right) < \inf_{i=1,\ldots,m} d\left(s_i^* s_i, s_i\right),$$
and that doesn't increase the asymptotic dimension
- Hence, $\operatorname{asdim}(S) = 0$ when S is locally finite

Local AF algebras and quasidiagonality I

Theorem (Chung, M. and Szakács - 22)

Let S be an inverse semigroup. TFAE:

(i) S is locally finite.
(ii) $\operatorname{asdim}(S, d) = 0$, where d is proper and r.inv.
(iii) $\ell^\infty(S) \rtimes_r S$ is local AF.
(iv) $\ell^\infty(S) \rtimes_r S$ is strongly quasidiagonal.

Local AF algebras and quasidiagonality I

Theorem (Chung, M. and Szakács - 22)

Let S be an inverse semigroup. TFAE:

(i) S is locally finite.
(ii) $\operatorname{asdim}(S, d) = 0$, where d is proper and r.inv.
(iii) $\ell^\infty(S) \rtimes_r S$ is local AF.
(iv) $\ell^\infty(S) \rtimes_r S$ is strongly quasidiagonal.

Remark: strongly quasidiagonal $\Rightarrow$ quasidiagonal

Local AF algebras and quasidiagonality I

Theorem (Chung, M. and Szakács - 22)

Let S be an inverse semigroup. TFAE:

(i) S is locally finite.
(ii) $\operatorname{asdim}(S, d) = 0$, where d is proper and r.inv.
(iii) $\ell^\infty(S) \rtimes_r S$ is local AF.
(iv) $\ell^\infty(S) \rtimes_r S$ is strongly quasidiagonal.

Remark: strongly quasidiagonal $\Rightarrow$ quasidiagonal

Theorem (Chung, M. and Szakács - 22)

Let S be an inverse semigroup. TFAE:

(i) S locally has finite components.
(ii) (S, d) is sparse, where d is proper and r.inv.
(iii) $\ell^\infty(S) \rtimes_r S$ is quasidiagonal.
(iv) $\ell^\infty(S) \rtimes_r S$ is finite.

A **bit** about the proof:

Local AF algebras and quasidiagonality II

A **bit** about the proof:

S locally finite $\Rightarrow \operatorname{asdym}(S) = 0$: sketched before

A **bit** about the proof:

$\underline{S \text{ locally finite} \Rightarrow \operatorname{asdym}(S) = 0}$: sketched before

$\underline{S \text{ locally finite} \Leftarrow \operatorname{asdym}(S) = 0}$:

Local AF algebras and quasidiagonality II

A **bit** about the proof:

S locally finite $\Rightarrow \operatorname{asdym}(S) = 0$: sketched before

S locally finite $\Leftarrow \operatorname{asdym}(S) = 0$:

S sparse $\Rightarrow \ell^\infty(S) \rtimes_r S$ is quasidiagonal:

Local finiteness vs. quasidiagonality

Sometimes these classes coincide, i.e.,

take $G \curvearrowright X$, where X is the Cantor set and G discrete group,

then: $\mathrm{Bis}(G \curvearrowright X)$ is locally finite $\Leftrightarrow$ $\mathrm{Bis}(G \curvearrowright X)$ is sparse.

Remark: these classes are, however, not the same!

- This division is impossible for groups, and
- already appeared in work of Li and Willett (2018)

Local finiteness vs. quasidiagonality

Sometimes these classes coincide, i.e.,

take $G \curvearrowright X$, where X is the Cantor set and G discrete group,

then: $\mathrm{Bis}(G \curvearrowright X)$ is locally finite $\Leftrightarrow$ $\mathrm{Bis}(G \curvearrowright X)$ is sparse.

Remark: these classes are, however, not the same!

- This division is impossible for groups, and
- already appeared in work of Li and Willett (2018)

Locally finite: direct limits of finite semigroups, and hence

fin. generated + locally finite $\Rightarrow$ finite

Local finiteness vs. quasidiagonality

Sometimes these classes coincide, i.e.,

take $G \curvearrowright X$, where X is the Cantor set and G discrete group,

then: $\mathrm{Bis}(G \curvearrowright X)$ is locally finite $\Leftrightarrow$ $\mathrm{Bis}(G \curvearrowright X)$ is sparse.

Remark: these classes are, however, not the same!

- This division is impossible for groups, and
- already appeared in work of Li and Willett (2018)

Locally finite: direct limits of finite semigroups, and hence

fin. generated + locally finite $\Rightarrow$ finite

Sparse: only have finite components (**not** uniformly), and hence

there are infinite sparse inverse semigroups, e.g., $S = \langle a \rangle$

Local finiteness vs. quasidiagonality

Sometimes these classes coincide, i.e.,

take $G \curvearrowright X$, where X is the Cantor set and G discrete group,

then: $\mathrm{Bis}(G \curvearrowright X)$ is locally finite $\Leftrightarrow$ $\mathrm{Bis}(G \curvearrowright X)$ is sparse.

Remark: these classes are, however, not the same!

- This division is impossible for groups, and
- already appeared in work of Li and Willett (2018)

Locally finite: direct limits of finite semigroups, and hence

fin. generated + locally finite $\Rightarrow$ finite

Sparse: only have finite components (**not** uniformly), and hence

there are infinite sparse inverse semigroups, e.g., $S = \langle a \rangle$

Thank you for your attention! Questions?