# Amenability and coarse geometry of (inverse) semigroups and C*-algebras

*by*


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These articles were presented by the author in the following talks:

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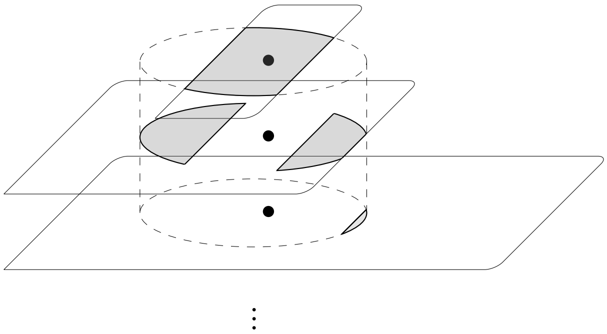
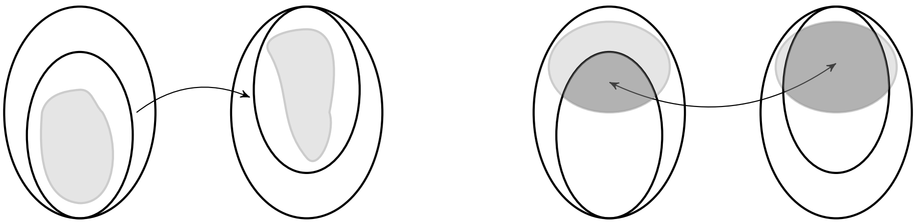
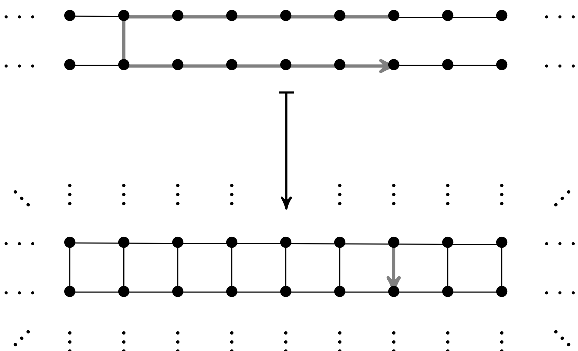
*A mis padres.*







*... sometimes it is right to doubt.*
William of Baskerville.

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# Chapter 1

# Introduction

It would be *utterly inadequate* to start this thesis by claiming it studies some deep connections between inverse semigroup actions, C*-algebras and metric spaces. On the other hand, it would also be *unjust* to commence by saying it is composed of some disconnected questions about semigroups and amenability, even though that might seem to be the case from time to time. It would, as well, be *boring* to start it with the classical story about the Lebesgue integral and the necessary existence of non-measurable sets, and how that fact irks us all. Nevertheless, it would certainly be *far from the case* to claim this thesis does not have its roots in that fact.

Therefore I am not going to start the thesis saying either of these things, and instead we will start with the following question:

*Do Ponzi schemes exist?*

By *Ponzi scheme* we mean a way of organizing people with the aim of creating money by moving it around. Ideally, Ponzi (the schemer) would convince Alice and Bob to give him 1\$, gaining a total net worth of 2\$. Alice and Bob would, in turn, each convince two other people, and both of them would gain a net worth of 1\$. Following this process *ad infinitum* every person in the scheme would, ideally, gain a net worth of, at least, 1\$, so that money has indeed been created. Of course, more important than the question above would be the next one:

*Can we create a working and real Ponzi scheme?*

As one can easily see, a *working* and *real* Ponzi scheme requires infinitely many people, so the immediate answer to the above question is, rather categorically, *no*. Going back to the first question, one acknowledges that a Ponzi scheme requires infinitely many people to even be possible, but not only does it require that. Indeed, $\mathbb{Z}$ has infinitely many elements, but in a rather straightforward composition:

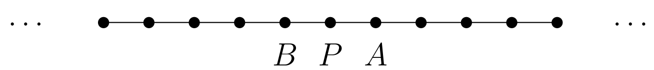


FIGURE 1.1: Ponzi tries to create a scheme in $\mathbb{Z}$, and fails miserably. The arrows (or, rather, lack thereof) represent the flow of the money.

Each vertex above represents a person, and two people are connected if they know each other. Needless to say, in this world knowing each other is a requirement to ask for money. The scheme starts with Ponzi, represented by vertex $P$, trying to convince Alice and Bob, $A$ and $B$, to give him 1\$. He fails, of course, for both Alice and Bob see through his lies and observe they will not be able to convince other people to give them money. Indeed, there is *no people to ask*, since, apart from Ponzi, Alice only knows one person (and the same goes for Bob). This makes it impossible to produce a Ponzi scheme with a $\mathbb{Z}$-like configuration. Observe that in this ideal world a person will not participate in the scheme unless they actually earn money as, for instance, neither Alice nor Bob will give Ponzi money in a $\mathbb{Z}$-like configuration.

Let us now rearrange the people in some other way:

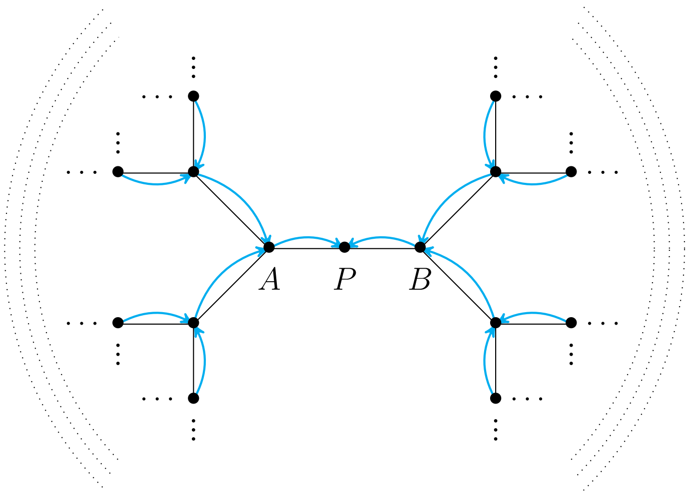


FIGURE 1.2: Ponzi tries to create a pyramid scheme in a tree, and succeeds. The arrows represent the flow of the money.

Ponzi now proposes a scheme to Alice and Bob. In this instance, both Alice and Bob realize the set of people they know is disjoint, and so is the set of people those people know (and so on). In particular, Alice and Bob realize it is in their best interest to follow the scheme. In the same fashion, so does everybody, and the scheme succeeds.

The difference between failing and succeeding is then the way people know each other. However, the fact that it *can* succeed is somehow *paradoxical*, for everyone within the system has gained money with only two actions (giving money and receiving it, both done only once). This is even more striking that *Hilbert's hotel*, where money (or rooms) can be created, but people have to be working for arbitrarily long periods of time in order to create arbitrarily large amounts of money (or rooms). Following this idea, we say Ponzi schemes are *paradoxical*.

Let us now study this *dynamical system* from the point of view of the money, and not the people. One can try to measure how much money a subset of people has with respect to the total amount of money of the system. Therefore, in a $\mathbb{Z}$-like configuration it is natural to say the people numbered by odd (or even) numbers amount to half the money of the system. However, if the Ponzi scheme succeeds this statement can no longer be made. Indeed, observe that the amount of money of the system remains equal by moving it around. Therefore, dividing the vertices of Figure 1.2 into $E$ and $O$, where $E$ is formed by the vertices at even distance of Ponzi (and analogously for $O$), one can see that $E$ has, at most, half the money $O$ has, and vicerversa. This is, of course, a contradiction. We may thus say that a $\mathbb{Z}$-like configuration is *meanable* (or *measurable*), as it has a *mean* (or a *measure*). However we shall instead say *amenable*, from its literal meaning and its similarity with the original *meanable*.

Lastly, one can instead choose to study the different configurations from the point of view of the connections, neither the people nor the money. In this sense, a $\mathbb{Z}$-configuration has the property that intervals have small boundary. Meanwhile, in a $\mathbb{F}_2$-configuration (as in Figure 1.2) balls around Ponzi have large boundaries, meaning that the people *in* a ball know a lot of people *out of* the ball. Actually, observe that, in order for the Ponzi scheme to work, the number of people on the boundary of the ball has to at least be of the same order as the number of people within the ball. We hence say that the $\mathbb{Z}$-configurations satisfy the *Følner condition*, while $\mathbb{F}_2$ does not.

These notions, namely *paradoxicality*, *amenability* and *Følner's condition*, can be properly defined. A natural question is whether these are equivalent:

> *Let $\mathcal{C}$ be a configuration of people. Is $\mathcal{C}$ paradoxical if, and only if, it is not amenable? Does it satisfy Følner's condition if, and only if, it is amenable?*

Not only is the answer as expected, the equivalence between these notions goes beyond Ponzi schemes and into groups, measure theory, metric spaces and C*-algebras.

This thesis, at its core, studies the relation between these notions, which lead to a *dichotomy*, within the context of semigroups, and builds from there and into related areas. Of particular interest, for various reasons, is the case of inverse semigroups, and their relation with C*-algebras, traces, metric spaces, property A and groupoids. Note all of these structures come naturally equipped with some *dynamics*, generalizing the idea of money going around in a Ponzi scheme.

## 1.1 Amenability in related areas

This section tries to explain how the notions above, namely *paradoxicality*, *amenability* and *Følner's condition*, appear not only in group theory but in related contexts. We will cite some of the most important works, but a full account of each work related to a subject will only be given when needed.

Historically speaking, the notion of amenability did not appear in relation with Ponzi schemes, but was coined by von Neumann [73] in an attempt to exclude the so-called *Banach-Tarski paradox* in **group theory** (see [92, 108] and references therein). In particular, von Neumann proved that a group being amenable was incompatible with a paradoxical behavior, which can be seen as a Ponzi scheme. After that, Tarski introduced in [106] a novel approach, namely the construction of a *type semigroup*, to prove that non-paradoxical groups are in fact amenable. Later Følner [39] gave another characterization of amenable groups, which was then generalized to semigroups, see [10, 29, 31]. Related to amenability, but not quite the same, is Yu's notion of *property A*, see [119] for the original work and [113] for an introduction to the topic.

The *paradoxical* behavior of a Ponzi scheme, or the existence of an *invariant measure*, can be replicated in the context of **metric spaces**. However, the simplest (and most natural) of the three notions to replicate is *Følner's condition*. Indeed, the idea of sets (mostly balls) having a *small boundary* is perfectly adaptable to metric spaces, yielding that a line satisfies the Følner condition (see [16, 44]). Again, observe that *small boundary* means that the set of points that are close to the original set is small with respect to the size of the original set. Likewise, a *measure* on a metric space is an assignment of weights to all subsets of the metric space (note this yields a *total* measure) in such a way that two sets in *bijective correspondence in a close manner* have the same measure. Following this idea, a metric space is *paradoxical* when it can be decomposed into two subsets in such a way that these sets are *close* to the original.

As may have been noticed, of utmost importance is the idea of *closeness*. For this we shall follow the notion of *closeness* from *coarse geometry* (see [76] for an introductory text to the subject), that is, two maps are *close* when one is a local perturbation of the other. Moreover, two sets are *close* if there is a map between them that is close to the identity map of either set. Note, for example, that the sets of even and odd numbers are close, while neither of them is close to the set of natural numbers. In this way, the difference between a Ponzi scheme and Hilbert's hotel, hinted at above, would be the same difference between two maps being close or not. Indeed, in a Ponzi scheme money can be created by a map that is close to the identity (see the blue arrows in Figure 1.2), while this is impossible in a $\mathbb{Z}$-like configuration, such as in Hilbert's hotel.

A particularly important observation within **operator algebra** literature is that amenability, as understood by von Neumann, Tarski and others, has a major role to play within the theory. In particular, the notion of amenability within operator algebras starts with the work of Connes [26], where he used amenability, adequately contextualized, to prove that *injectivity implies hyperfiniteness*. This way, if a C*-algebra is represented in a Hilbert space $\mathcal{H}$, we say it satisfies the *Følner condition* whenever one can find finite-dimensional subspaces of $\mathcal{H}$ that are *almost* invariant under the representation. Observe that, within this context, *almost* may mean various things:

- On the one hand, it can mean that most of the subspace remains close to the subspace, and only an $\varepsilon$-part of it is allowed to go far from it. This approach, which was Connes' original one, yields *hypertracial* C*-algebras and *amenable traces*, and was followed by, for instance, Bédos [14, 15] (see also [20, Chapter 6]). Observe that, within this context, an *amenable trace* serves the purpose of an invariant mean. Likewise, a *paradoxical decomposition* of the algebra is then a decomposition of the unit by isometries with orthogonal ranges, which, in a sense, exactly mimics the paradoxical behavior of a group.

- On the other hand, it can mean every element of the subspace goes to somewhere that is $\varepsilon$-close to the subspace. Note that this is different from the above, since no element in the subspace is allowed to go far from it, no matter how large the subspace itself is. This approach yields *quasi-diagonal* C*-algebras, following the analogous operator notion by Halmos [47]. This class of C*-algebras was later studied by Hadwin and Rosenberg [46] (among many others) and characterized by Voiculescu [111, 112]. This notion, although more restrictive, is much more useful than the one above, and was key in the so-called *classification program*, particularly by the work of Tikuisis, White and Winter [107] (see also Gabe's [40] and Schafhauser's [96]).

Lastly, the notions of amenability can also be translated into **(inverse) semigroups**, following the work of Day [29]. Within this setting, a measure on the semigroup is said to be invariant whenever it assigns the same number to the preimage of a set and the set itself. Moreover, as we shall see, within this generality there is no clear notion of *Følner's condition* (see, e.g., [31, 71]), nor is there any clear candidate of *paradoxicality.* However, one of the main points of the thesis is to prove that the dichotomy can be recovered should one restrict to the case of *inverse semigroups.*

## 1.2 Goal and main questions of the thesis

These final sections of the introduction aim to state the main questions of the thesis, spoil some main results and sketch the structure on which it is based on. For necessary reasons we give the answers to the questions themselves, so as to justify the next questions we are interested in.

Our work starts in the context of general semigroup theory, and seeks to replicate the classical equivalence between *paradoxicality*, *amenability* and the *Følner's condition*:

(QS.1) *Can one generalize the notions of Følner sequence and paradoxical decomposition to the context of semigroups in such a way that the classical dichotomy amenable vs. paradoxical remains true?*

(QS.2) *If the answer to the above is negative, is there any particular class of semigroups where such dichotomy may be found?*

As can be expected, the answer to (QS.1) is negative, and the answer to (QS.2) is positive. Of particular interest is the class of *inverse semigroups*, these being to groups what partial isometries are to unitary operators. In this context, the following are only natural.

(QI.1) *In the particular class of inverse semigroups, what form (or forms) does the classical dichotomy amenable vs. paradoxical take?*

(QI.2) *Can we build a C*-algebra out of an inverse semigroup that inherits the amenability of the semigroup? Does this algebra have traces? Can these traces be seen as some finitely additive probability measures in the semigroup itself?*

(QI.3) *Are the traces above always amenable? That is, can they be approximated by traces in matrices in 2-norm?*

The C*-algebra we refer to in (QI.2) is the analogous to the *uniform Roe algebra* of a group and, as we shall see, its traces can be seen as a particular class of finitely additive probability measures in the semigroup itself. Moreover, the fact that the natural C*-algebra appearing is close (in nature) to being a *uniform Roe algebra* naturally leads one to wonder whether there is some metric structure behind the semigroup. This, for instance, holds for discrete and countable groups, where one can equip them with an essentially unique metric yielding the same C*-algebra. Hence the following geometric questions arise:

(QM.1) *Can we equip an inverse semigroup with a metric that is suitably proper and right-invariant and generalizes the path metric in the Cayley graph of a finitely generated group?*

(QM.2) *If the answer to the above is positive, does it yield a metric space whose associated uniform Roe algebra is precisely the C*-algebra considered in (QI.2)?*

(QM.3) *Are there properties of these spaces that are quasi-isometric invariants? In particular, can amenability of the semigroup be seen metrically? Does the property A of the space transform into a C*-property?*

The text completely answers these questions, though the answers themselves may be surprising. For instance, the answer to (QM.2) is *sometimes*. Dealing with metric spaces and inverse semigroups one eventually, and inevitably, arrives to questions about grouopids. In particular, how these notions can be seen in the context of a certain groupoid associated to the inverse semigroup.

(QG.1) *Can the amenability of an inverse semigroup be related to the amenability of its universal groupoid? What about its property A?*

Lastly, one could wish to improve (QI.3), in the sense of using norm approximations instead of 2-norm approximations. Indeed, the work of Tikuisis, White and Winter [107] in quasi-diagonal approximations naturally arises, and begs the question of whether the C*-algebras considered so far are quasi-diagonal.

(QQ.1) *Does the analog of Rosenberg's conjecture for inverse semigroups hold? That is, is the reduced semigroup C*-algebra quasi-diagonal whenever the semigroup is amenable? What about the converse?*

(QQ.2) *Under which circumstances is the reduced semigroup C*-algebra stably finite? Is this the only obstruction to quasi-diagonality in the context of inverse semigroups? If not, does amenability play a role?*

All of these questions are either answered or addressed in the text, along with some other inquiries that arise along the way. It should be said that, among these, the most important question is the rather innocent-looking question (QI.1). From it and its answer everything else derives, for its answer sketches the main ideas behind everything else, stating in a precise manner the language that ought to be used.

## 1.3 Brief summary of main results

We first investigate some of the most common notions of amenability (following Day's approach) in the context of general semigroups, coming to the conclusion that there is no satisfactory answer (see Section 3.1). In the subsequent Chapter 4 we restrict to the case of inverse semigroups (and their representations), and prove that Day's amenability is equivalent to the conjunction of two independent conditions, namely *localization* and *domain-measurability* (see Theorem 4.1.3). Following this approach, we then study the trace space of a certain uniform Roe algebra associated to the inverse semigroup, and relate these traces with certain measures in the semigroup (see Theorem 4.3.7).

The proof of Theorem 4.3.7, in particular a Følner-like characterization, provides the necessary intuition in order to consider an inverse semigroup in a geometric manner. In this fashion, in Section 5.1.1 we introduce a distance in the semigroup, and observe that it generalizes the path distance in the Cayley graph of a group. Using this distance, we then give a condition in the semigroup that characterizes when the C*-algebra considered in Chapter 4 is actually a uniform Roe algebra (see Theorem 5.2.23). Moreover, we also tackle the study of some quasi-invariant conditions, such as amenability (see Theorem 5.3.5) and property A (see Theorem 5.3.10).

Lastly, Chapter 6 studies the relation between the amenability of the semigroup and the quasi-diagonality of its reduced C*-algebra, that is, studies the so-called *Rosenberg's conjecture* in the context of inverse semigroups.

## 1.4 Structure of the thesis

Figure 1.3 depicts the structure and flow of the thesis. As can be deduced from the picture, the thesis is organized around the two main bodies of content, namely Chapters 4 and 5.

The text starts, as is only natural, with a not-so-brief introduction to all the mathematical background needed later, which will be covered in Chapter 2. In particular, Chapter 2 includes notions ranging from group theory to C*-algebras and metric space theory.

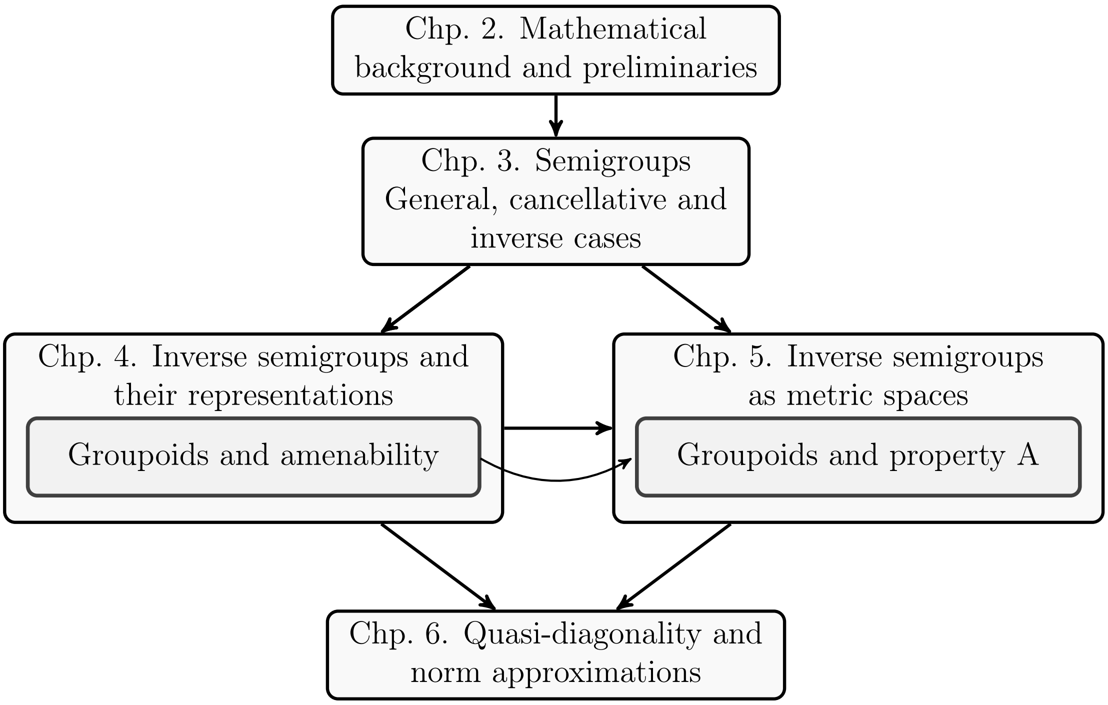


FIGURE 1.3: Rough structure and flow of the thesis.

Chapter 3 then covers all of the aspects about general semigroup theory. In particular, it tries to answer question (QS.1), whose answer is negative, and the subsequent (QS.2), whose answer is positive. For lack of a better place, Chapter 3 also covers the inverse semigroup theory needed later, since this is the context the main part of the thesis is based upon.

Then Chapter 4 studies some of the main questions of the thesis, namely questions (QI.1), (QI.2) and (QI.3). These are answered in a positive manner via the probably most important result of the thesis, see Theorem 4.1.3. In a way, this serves as a starting point, clearly stating the language and technicalities we shall have to deal with later. Hence most of the other definitions naturally emerge from the so-called *domain-measurability* of a *representation*. In addition, of particular interest in this Chapter 4 are what have been called 2-*norm approximations*, that is, approximations of a C*-algebra using a trace norm instead of a uniform norm. As we have mentioned, these approximations, as virtually every other result in the text, is an easy corollary of the answer to (QI.1), namely Theorem 4.1.3.

The research developed in Chapter 4 raises some natural intuitions and questions. In particular, even though Chapter 4 is *analytical*, it does raise the question about whether there is some *geometric meaning* behind the analysis developed. This intuition, properly speaking, is then developed in Chapter 5, and answers positively questions (QM.1), (QM.2) and (QM.3). Hand-waving, Chapter 5 discusses the fact that inverse semigroups, just as groups, can be equipped with a natural metric, yielding interesting geometries. Moreover, these metric spaces inherit some properties of the semigroups themselves, such as *domain-measurability* or *Yu's property A*.

An important feature of Chapters 4 and 5 is the fact that both include a section with examples and a section about the relation of that chapter with some notions of groupoid theory (see Figure 1.3). These will be covered in Sections 4.5 and 5.5, respectively. In addition, these sections should be seen as only one unit that has been partitioned for the sake of inner coherence within the thesis.

Lastly, going back to the 2-norm approximation appearing in (QI.3) and Chapter 4, the final Chapter 6 studies whether one can obtain a *quasi-diagonal* approximation, answering questions (QQ.1) and (QQ.2) partially.

The thesis ends with Chapter 7, asking some other questions that have been left unanswered along the text, or have appeared in the study.

*Never trust these people, Adso.*
*They always manage to respect the letter*
*and violate the contents of a deal.*
William of Baskerville

# Chapter 2
# Mathematical background and preliminaries

The goal of this section is to set the main mathematical background needed in the thesis. Everything presented in this section is already known, and most is classical work. It thus goes without proof unless it is convenient for the *narrative* of the thesis to sketch the argument. Moreover, due to the nature of the thesis itself, this chapter has to introduce numerous topics. Therefore, and in order to keep it at a reasonable length, I have decided to only introduce the topics, and reference some nice introductory texts to each subject at the beginning of the corresponding section. This, in addition, facilitates the writing of the chapter. It should also be mentioned that any reader familiar with groups, algebras, metric spaces, operator algebras or semigroups might skip the corresponding section. Lastly, everything presented henceforth shall be considered known in the upcoming chapters.

## 2.1 Groups and algebras

All groups are supposed to be countable and discrete, though not necessarily finitely generated. Likewise, all algebras are over $\mathbb{C}$, even though one can work with more general fields (see, e.g., [13]) and of countable dimension.

### 2.1.1 Amenable groups

As was mentioned in the introduction, amenability was introduced by von Neumann as a condition incompatible with the paradoxical behavior of the so-called *Banach-Tarski paradox* (see [73, 106, 92, 108]). Since then, it has been extensively studied, first within the context of group (and semigroup) theory (see [29, 71, 10, 31]), and then in related areas, such as algebras (see [13, 22, 7]), metric spaces (see [21, 8]) and operator algebras (see [26, 20]). In this section we explore the notion of amenability in the group context and, more generally, in the context of algebras over $\mathbb{C}$. We first introduce the three notions that play a main role within the thesis.

**Definition 2.1.1.** Let $G$ be a countable and discrete group.

(1) $G$ is (left) *amenable* if there exists a (left) invariant measure on $G$, i.e., a finitely additive probability measure $\mu\colon \mathcal{P}(G) \to [0,1]$ such that $\mu(g^{-1}A) = \mu(A)$ for all $g \in G$ and $A \subset G$.

(2) $G$ satisfies the *Følner condition* if for every $\varepsilon > 0$ and finite $\mathcal{F} \subset G$ there is a finite non-empty $F \subset G$ such that $|gF \cup F| \leq (1+\varepsilon)|F|$ for every $g \in \mathcal{F}$. In such case we say that $F$ is $(\mathcal{F},\varepsilon)$-*invariant*, or that it is a $(\mathcal{F},\varepsilon)$-*Følner set*.

(3) $G$ is *paradoxical* if there are sets $A_i, B_j \subset G$ and elements $a_i, b_j \in G$ such that

$$\begin{aligned} G &= a_1A_1 \sqcup \cdots \sqcup a_nA_n = b_1B_1 \sqcup \cdots \sqcup b_mB_m \\ &\supset A_1 \sqcup \cdots \sqcup A_n \sqcup B_1 \sqcup \cdots \sqcup B_m. \end{aligned}$$

We call such a partition a *paradoxical decomposition of* $G$.

The three notions above shall drive a good portion of this thesis, namely most of Chapter 4, and their importance and relation will be clear by then. First of all, however, we need to introduce some known results, the first of which gives an alternative (and useful) characterization of invariant measures.

**Lemma 2.1.2.** *A group is amenable if, and only if, there is a $G$-invariant mean on $\ell^\infty(G)$, that is, a normalized linear functional $m\colon \ell^\infty(G) \to \mathbb{C}$ such that $m(gf) = m(f)$, for all $f \in \ell^\infty(G)$ and $g \in G$, where $(gf)(h) := f(g^{-1}h)$.*

*Proof.* The proof is well known. Indeed, the map $m \mapsto \mu_m(A) := m(p_A)$, where $p_A \in \ell^\infty(G)$ is the characteristic function of $A$, can be proven to be a bijective correspondence between the set of invariant measures and the set of invariant means. □

The second lemma proves that the Følner sets as in Defintion 2.1.1 (2) may be taken to contain any given finite subset of $G$. This, for our purposes, proves that, in the group case, the Følner condition and the *proper* Følner condition[1] are equivalent (see this in stark contrast to Lemma 2.1.9 in the context of algebras).

**Lemma 2.1.3.** *Let $G$ be a countable group satisfying the Følner condition. Then for any $\varepsilon > 0$ and any finite $A, \mathcal{F} \subset G$ there is some $F \subset G$ such that $F$ is $(\mathcal{F},\varepsilon)$-invariant and $A \subset F$.*

*Proof.* We only sketch the proof, since it will be properly reproduced in Theorem 3.1.1 in the more general context of semigroups. Observe the set of pairs $\{(\mathcal{F},\varepsilon) \mid \mathcal{F} \subset G \text{ and } \varepsilon > 0\}$ is partially ordered by inclusion in the first coordinate and by greater or equal in the second coordinate. Therefore as $\varepsilon$ decreases $|F|$ increases. There are then two cases, either the cardinality of $F$ is bounded or not. On the one hand, if $|F|$ is bounded then we get that $gF = F$ for every $g \in G$, and, *a posteriori*, we conclude that $G = F$ is a finite group. On the other hand, if $|F|$ grows arbitrarily we may, without loss of generality, suppose that $|F| \geq |A|/\varepsilon$. The set $F \cup A$ is then a Følner set containing $A$. In either case, the conclusion of the lemma holds. □

[1]The *proper* Følner sequence has not been defined yet, but it boils down to the statement of Lemma 2.1.3.

Finally, the following well known lemma introduces the idea of a *Følner sequence* of a group $G$. It is, of course, equivalent to the Følner condition in Definition 2.1.1 (2). Its proof is routine. Moreover, observe that a version of the below lemma is true for arbitrary discrete groups so long as one considers nets instead of sequences.

**Lemma 2.1.4.** *Let $G$ be a countable group. Then $G$ satisfies the Følner condition if and only if there is a sequence $\{F_n\}_{n\in\mathbb{N}}$ of finite non-empty $F_n \subset G$ such that*

$$|gF_n \cup F_n| / |F_n| \xrightarrow{n\to\infty} 1$$

*for every $g \in G$. In particular, by Lemma 2.1.3, one may choose $\{F_n\}_{n\in\mathbb{N}}$ to be increasing and exhaustive, i.e., $F_n \subset F_{n+1}$ and $\cup_{n\in\mathbb{N}} F_n = G$.*

We now wish to give a myriad of characterizations of the amenability of a group (see Theorem 2.1.5 below). To that end, we recall the following customary notations. $\mathbb{C}G$ is the *group algebra of* $G$, that is, the algebra formed by formal finite sums $\sum a_g g$, where $a_g \in \mathbb{C}$ and $g \in G$, equipped with the product coming from extending the product of $G$. If closed in the spatial norm it yields $C_r^*(G)$, i.e., the *reduced $C^*$-algebra of* $G$. Meanwhile, if one closes in the maximal norm then it yields $C^*(G)$, the *full $C^*$-algebra of* $G$. Moreover, representing $C_r^*(G)$ canonically in $\mathcal{B}(\ell^2 G)$ and closing it in the weak operator topology produces $L(G)$, which denotes the *von Neumann algebra of* $G$. Finally, $C_u^*(G)$ stands for the *uniform Roe algebra of* $G$. The theorem below does not pretend to be a complete enumeration of all the equivalent characterization of the amenability of a group, since that task is almost impossible by now (and we want to keep the text to a finite length). We, however, collect the most important characterizations from the point of view of this thesis.

**Theorem 2.1.5.** *Let $G$ be a countable and discrete group. The following assertions are then equivalent.*

*(1) $G$ is amenable (see Definition 2.1.1 (1)).*
*(2) $G$ satisfies the Følner condition (see Definition 2.1.1 (2)).*
*(3) $G$ is not paradoxical (see Definition 2.1.1 (3)).*
*(4) $\mathbb{C}G$ is algebraically amenable (see Definition 2.1.7).*
*(5) $C_r^*(G)$ is nuclear (see Definition 2.3.1).*
*(6) $C_r^*(G)$ and $C^*(G)$ are $^*$-isomorphic.*
*(7) $C_r^*(G)$ is quasi-diagonal (see Definition 2.3.10).*
*(8) $C_r^*(G)$ embeds into $\mathcal{Q}$, the universal UHF algebra.*
*(9) $L(G)$ embeds into $\mathcal{R}$, the hyperfinite $II_1$ factor.*
*(10) $C_u^*(G)$ admits a, necessarily amenable, trace (see Definition 2.3.7).*
*(11) $C_u^*(G)$ is not properly infinite (see Definition 2.3.7).*
*(12) $[0] \neq [1]$ in the $K_0$-group of $C_u^*(G)$.*

*Proof.* The proof relies on results of many hands and, thus, and in order to keep it short, we only cite the papers where each statement was proved, i.e., proving that amenability and non-paradoxicality are equivalent notions for groups.

Regarding the *algebraic* characterizations, (1) $\Leftrightarrow$ (2) is classical, and is due to Følner in [39]. Tarski proved (1) $\Leftrightarrow$ (3) in [106]. (1) $\Leftrightarrow$ (4) was proven in [34] (see also [13]).

The C*-characterizations began with the work of Connes [26], where he essentially showed (1) $\Leftrightarrow$ (9). The equivalence (2) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) follows from work of Reiter and the fact that if $G$ is amenable then the trivial representation is *weakly contained* in the regular representation (see [20, Theorem 2.6.8]). More recently, Tikuisis, White and Winter proved (1) $\Leftrightarrow$ (7) as a corollary to the main theorem of [107], while (1) $\Leftrightarrow$ (8) was proved by Schafhauser in [97]. Finally, (10) and (11) are well known to be equivalent to $G$ being amenable, see [91] for a proof (see also [8] for the analogue proof for metric spaces). □

**Remark 2.1.6.** There are two main strategies to prove the equivalence between (1) and (3).

(i) On the one hand one can use Zorn's Lemma and a doubling argument to produce a paradoxical decomposition from a non-Følner group, as is done in [52, Theorem 4.4] (see [7, Theorem 2.17] for the metric space scenario and [7, Theorem 4.6] for a linearization of this technique to the case of algebras).

(ii) On the other hand one may construct the so-called *type semigroup* of the group (or, more generally, the action of the group). This, in turn, allows to construct an invariant measure on a non-paradoxical group $G$, which was the original strategy followed by Tarski in [106]. Moreover, and historically speaking, this strategy has been used more. Indeed, the $K_0$-group of a C*-algebra (or, rather, an intermediate step in its construction) may be seen as a type semigroup (see [86, 87]), and so may the Cuntz semigroup (see, e.g., [4] and references therein). In general, and hand-waving, any semigroup resulting from an equivalence relation given by an action of a group is, in essence, a type semigroup. In the case of this thesis we will *follow the trend* and use this strategy (see Section 4.2.1).

Using Theorem 2.1.5 we can study the class of amenable groups up to a reasonably detailed point. It is well known (and straightforward to prove) that both finite and abelian groups are amenable. Moreover, amenability is preserved by taking subgroups, short exact sequences and direct limits. Finally, free non-abelian groups are non-amenable, as is every group with free non-abelian subgroups (see, e.g., [23, Chapter 4]).

### 2.1.2 Algebraic amenability

Algebraic amenability was introduced in [44] (see also [34]) as an analogue of amenability in the context of associative algebras. It follows the Følner characterization present in Definition 2.1.1, but uses finite dimensional subspaces of the algebra instead of subsets of the group.

**Definition 2.1.7.** Let $\mathcal{A}$ be an associative algebra of countable dimension.

(1) We say $\mathcal{A}$ is *algebraically amenable* if for every $\varepsilon > 0$ and finite $\mathcal{F} \subset \mathcal{A}$ there is a non-zero finite dimensional subspace $W \leq \mathcal{A}$ such that $\dim(AW + W) \leq (1+\varepsilon)\dim(W)$ for every $A \in \mathcal{F}$. In this case, we say that $W$ is $(\mathcal{F}, \varepsilon)$*-invariant*, or that it is $(\mathcal{F}, \varepsilon)$*-Følner*.

(2) We say $\mathcal{A}$ is *properly algebraically amenable* if for every $\varepsilon > 0$ and every finite $\mathcal{F}, A \subset \mathcal{A}$ there is a finite dimensional $W \leq \mathcal{A}$ such that $A \subset W$ and $W$ is $(\mathcal{F}, \varepsilon)$-invariant.

Observe that the definition above translates the Følner condition of groups into the context of algebras. Likewise, one could also define the analogous notions of invariant mean or paradoxical decomposition. However, as we won't delve much into those, we will skip those definitions (see [7] for a detailed account of these).

The class of algebraically amenable algebras is, as was the class of amenable groups, plentiful. It contains every abelian algebra, every finite-dimensional algebra, and is preserved under various constructions (see [34, 7]). There is, however, a key difference, namely in the relation between the proper Følner condition and the non-proper Følner condition. The following example and lemma exactly characterize that difference (see this in contrast to Lemma 2.1.3).

**Example 2.1.8.** Let $\mathcal{A}$ be an associative algebra, and let $I \leq \mathcal{A}$ be a non-zero left ideal of finite dimension. Then $\mathcal{A}$ is algebraically amenable, since $aI \subset I$ for all $a \in \mathcal{A}$. Therefore $\mathcal{A}I \subset I$ and

$$\dim(\mathcal{A}I + I) \leq \dim(I),$$

proving that $I$ is a $(\mathcal{F}, \varepsilon)$-invariant subspace for every $\mathcal{F} \subset \mathcal{A}$ and $\varepsilon > 0$. Nevertheless, $\mathcal{A}$ might not be properly algebraically amenable. Indeed, take $\mathcal{A} := \mathbb{C} \oplus \mathbb{C}\mathbb{F}_2$. Then $I := \mathbb{C} \oplus 0$ is a left ideal of finite dimension, but $\mathbb{C}\mathbb{F}_2$ is not algebraically amenable (see Theorem 2.1.5), and therefore $\mathcal{A}$ itself is algebraically amenable but not properly algebraically amenable.

**Lemma 2.1.9.** *Let $\mathcal{A}$ be an algebra of countable dimension. Suppose $\mathcal{A}$ is algebraically amenable but not properly algebraically amenable. Then there is some non-zero element $a \in \mathcal{A}$ such that $\mathcal{A} \cdot a$ is of finite dimension.*

*Proof.* As in Lemma 2.1.3, we only sketch the proof, so for a complete proof see [7, Theorem 3.9] (see also Theorem 3.1.1 below). Since $\mathcal{A}$ is not properly algebraically amenable one can prove there are $(\mathcal{F}, 0)$-invariant subspaces of $\mathcal{A}$. Indeed, this follows from the fact that the set of pairs $\{(\mathcal{F}, \varepsilon)\}$ is partially ordered and the fact that the dimension of any subspace of $\mathcal{A}$ is a positive integer. Therefore, if a certain subspace is $(\mathcal{F}, \varepsilon)$-invariant for a sufficiently small $\varepsilon$, then it must actually be $(\mathcal{F}, 0)$-invariant. From this, and taking larger and larger sets $\mathcal{F} \subset \mathcal{A}$, one can prove there is some subspace $W$ of finite dimension such that $\mathcal{A}W \subset W$. Hence any $a \in W$ will satisfy the conditions in the statement. □

## 2.2 Metric spaces

This section aims to introduce some notions about (extended) metric spaces needed later in the thesis. Just as the previous section, we only state some basic definitions and results regarding coarse geometry (see Section 2.2.1), amenability and property A of metric spaces (see Section 2.2.2) and groups as metric spaces (see Section 2.2.3). As general references see [44, 76, 113] for introductory texts to metric spaces of bounded geometry, coarse geometry and property A. In particular, see [76, Section 1.2] for an introduction to groups as metric spaces.

For reasons that will be clear in Chapter 5 we need to deal with *extended* metric spaces. These are metric spaces where two points might be at infinite distance:

**Definition 2.2.1.** A set $X$ equipped with a function $d: X \times X \to [0, \infty]$ is an *extended metric space* if $d(x,x) = 0$ for all $x \in X$, $d$ is symmetric and it satisfies the triangle inequality, that is, for every $x, y, z \in X$

$$d(x,z) \leq d(x,y) + d(y,z),$$

where we use the natural convention that $\infty + d = d + \infty = \infty$ for any $d \in [0, \infty]$.

The technical issues arising from considering extended metric spaces instead of usual metric spaces can be easily overcome in the context of this thesis. Indeed, note that if $(X,d)$ is an extended metric space then it decomposes into its *coarse connected* components[2], i.e., $X = \sqcup_{i \in I} X_i$, where $(X_i, d|_{X_i})$ is a usual metric space. Moreover, we aim to study some *large-scale* properties of $X$ and, in this way, whenever we say $(X,d)$ has property $\mathfrak{P}$ we mean to say that some (or every, depending on the context) $X_i$ satisfies property $\mathfrak{P}$. This discussion shall be clear when we get to property A (see Definition 2.2.12).

For now we introduce some customary definitions and notations. Given a point $x \in X$ and a non-negative $r \geq 0$ we denote the ball of radius $r$ centered around $x$ by $B_r(x)$. The following are well known (see, e.g., [76, Definition 1.2.6]).

**Definition 2.2.2.** Let $(X,d)$ be an extended metric space.

(1) We say $(X,d)$ is *uniformly discrete* if $\inf_{x,y \in X} d(x,y) > 0$.

(2) We say $(X,d)$ is *locally finite* if it is uniformly discrete and $B_r(x)$ is a finite set for every $r \geq 0$ and $x \in X$.

[2] Throughout the thesis we simply call them components.

(3) We say $(X, d)$ has *bounded geometry* if it is locally finite and for every $r \geq 0$ there is some $m \geq 0$ such that $\sup_{x \in X} |B_r(x)| \leq m$, that is, the $r$-balls of $X$ are of uniformly bounded cardinality.

The class of metric spaces of bounded geometry encompasses many examples. For instance, every countable group can be equipped with a unique (up to *coarse equivalence*) proper and right invariant metric (see Section 2.2.3), yielding a unique (up to *coarse equivalence*) metric space of bounded geometry. With a different approach, every undirected graph may be equipped with the usual path length metric, and the resulting metric space is of bounded geometry if and only if the original graph has uniformly bounded valences. There are, however, metric spaces that do not come from graphs and still are of bounded geometry, such as $X := \{n^2 \mid n \in \mathbb{N}\}$ equipped with the metric inherited from $\mathbb{N}$.

Finally, recall (see, e.g., [76, Definition 1.1.11]) that we say a metric space $X$ is *geodesic* if for any $x, y \in X$ there is an isometric embedding $\gamma\colon [0, d(x,y)] \to X$ such that $\gamma(0) = x$ and $\gamma(d(x,y)) = y$. The image of such a $\gamma$ is naturally called a *geodesic between $x$ and $y$*. The prime example of geodesic space, for the purposes of this thesis, is an undirected graph equipped with the usual path metric. In this case, the embedding $\gamma\colon [0, d(x,y)] \to X$ can be constructed in the obvious way.

### 2.2.1 Coarse geometry

Coarse geometry is the study of metric spaces *from far away*. It has been extensively studied in the literature (see [89, 113, 76] for nice expositions on this topic), and, in particular, within geometric group theory (see Section 2.2.3).

Observe that, even within the context of metric spaces of bounded geometry, given any bounded geometry space $(X, d)$ there are numerous metric spaces that *look like $X$ from far away*. The two following definitions try to mimic this idea of observing a metric space from far away, and both will yield equivalence relations in the class of metric spaces of bounded geometry. Even though within the literature one usually studies non-extended metric spaces, for the purposes of this thesis we have to deal with infinite distances, and must therefore slightly adapt the notions.

**Definition 2.2.3.** Let $(X, d_X)$ and $(Y, d_Y)$ be two extended metric spaces, and let $\phi\colon X \to Y$ be a map between them.

(1) Given $l \geq 1$ and $c \geq 0$ we say that $\phi$ is an $(l,c)$*-quasi-embedding*, or *quasi-isometric embedding*, if for every $x_1, x_2 \in X$

$$l^{-1} d_X(x_1, x_2) - c \leq d_Y(\phi(x_1), \phi(x_2)) \leq l\, d_X(x_1, x_2) + c.$$

(2) Given $r \geq 0$ we say $\phi$ is $r$*-quasi-surjective* if $d_Y(y, \phi(X)) \leq r$ for every $y \in Y$.

(3) We say that $\phi$ is a *quasi-isometry* if it is an $(l,c)$-quasi-embedding and $r$-quasi-surjective for some $l, r, c$. Likewise, we say that $X$ and $Y$ are *quasi-isometric* if there is a quasi-isometry between them.

Note that in (1) above we are implicitly using the natural convention that $l^{\pm 1}\cdot\infty \pm c = \infty$. This means that whenever $d_X(x_1,x_2) = \infty$ (i.e., $x_1$ and $x_2$ belong to different components of $X$), then $d_Y(\phi(x_1),\phi(x_2)) = \infty$. Equivalently, a quasi-embedding of $X$ into $Y$ must respect the components of $X$. Moreover, if we consider more general control functions in the bounds of Definition 2.2.3 (1) we produce the notion of *coarse equivalence* (see [76, Section 1.4]).

**Definition 2.2.4.** Let $(X,d_X)$ and $(Y,d_Y)$ be two extended metric spaces, and let $\phi\colon X\to Y$ be a map between them.

(1) We say that $\phi$ is a *coarse embedding* if there are functions $\rho_-,\rho_+\colon[0,\infty]\to[0,\infty]$ such that
$$\rho_\pm(r)\xrightarrow{r\to\infty}\infty \text{ and } \rho_\pm(\infty)=\infty$$
and such that for every $x_1,x_2\in X$
$$\rho_-(d_X(x_1,x_2))\leq d_Y(\phi(x_1),\phi(x_2))\leq\rho_+(d_X(x_1,x_2)).$$

(2) We say that $\phi$ is a *coarse equivalence* if it is a quasi-surjective coarse embedding. Likewise, we say that $X$ and $Y$ are *coarsely equivalent* if there is a coarse equivalence between them.

It follows from the definitions that quasi-isometric spaces must also be coarsely equivalent. The reverse implication, however, is not true in general. Indeed, $\{n^2\}_{n\in\mathbb{N}}$ and $\{n^3\}_{n\in\mathbb{N}}$ are coarsely equivalent but are not quasi-isometric, as is easily proven. Nevertheless, the two conditions are actually equivalent in a large class of examples, including the examples coming from finitely generated semigroups, which will be the ones we shall be interested in. See [76, Theorem 1.4.13] for a proof of the following.

**Proposition 2.2.5.** *Let $X$ and $Y$ be metric spaces, and let $\phi\colon X\to Y$ be a coarse equivalence. If $X$ is quasi-isometric to a geodesic space, then $\phi$ is actually a quasi-isometry. In particular, the class of quasi-isometries and the class of coarse equivalences between undirected graphs are the same.*

Recall, as well, the notion of *coarse disjoint union* (see [76, Example 1.4.12]).

**Definition 2.2.6.** Let $\{(X_i,d_i)\}_{i\in I}$ be a collection of metric spaces. A *coarse disjoint union* of $\{(X_i,d_i)\}_{i\in I}$ is the set $X=\sqcup_{i\in I}X_i$ equipped with any metric $d$ extending every $d_i$ and such that
$$d(X_i,X_j)\xrightarrow{i,j\to\infty}\infty,$$
that is, for every $p\in\mathbb{N}$ there is some finite $K\subset I$ such that if $i,j\notin K$ and $i\neq j$ then $d(X_i,X_j)\geq p$.

Observe that we say *a* instead of *the* coarse disjoint union since there may be more than one metric as above, and therefore $X$ is not uniquely determined. For instance, the metric space $\{n^2\}_{n\in\mathbb{N}}$ equipped with the metric coming from $\mathbb{N}$ is a coarse disjoint union of countably many points.

### 2.2.2 Amenability and property A

As was the case for groups (see Section 2.1.1), one can say when a metric space is *amenable*, satisfies the *Følner condition* or is *paradoxical*. The first of these definitions appeared in [44], and is based on the idea of a Følner set being a set with small *boundary*. The others appeared in later work, for instance see [21, 8] for a detailed study of their relation. Recall that the *r-boundary* of a subset $A \subset X$ is

$$\partial_r A := \{x \in X \mid d(x, A) \leq r \text{ and } d(x, X \setminus A) \leq r\},$$

that is, $\partial_r A$ is the set of points close to $A$ and to its complement. Analogously one may define the *r-neighborhood* of a set $A \subset X$ as

$$\mathcal{N}_r A := \{x \in X \mid d(x, A) \leq r\}.$$

It is clear that $\partial_r A \subset \mathcal{N}_r A$.

Note that metric spaces are *static* objects and, hence, we need to introduce some dynamics in them to be able to say when a metric space is *amenable*. In this case, the dynamics are given by so-called *partial translations*: given $A, B \subset X$, we say a bijection $t: A \to B$ is a *partial translation* when

$$\sup_{a \in A} d(a, t(a)) < \infty,$$

that is, when $t$ is isometric up to a uniformly local perturbation. With this at hand, we can now translate the notion of invariant measure to the setting of metric spaces.

**Definition 2.2.7.** Let $(X, d)$ be an extended metric space.

(1) An *invariant measure* on $X$ is a finitely additive probability measure $\mu$ on $X$ such that $\mu(A) = \mu(B)$ for every partial translation $t: A \to B$.

(2) $X$ satisfies the *Følner condition* if for every $r, \varepsilon > 0$ there is some finite non-empty $F \subset X$ with small $r$-neighborhood, that is, $|\mathcal{N}_r F| \leq (1 + \varepsilon)|F|$.

We say $X$ satisfies the *proper Følner condition* if for every $n, r, \varepsilon > 0$ there is some finite $(r, \varepsilon)$-Følner set as above such that $|F| \geq n$.

(3) A *paradoxical decomposition* of $X$ is a partition $X = X_1 \sqcup X_2$ with partial translations $t_i: X \to X_i$.

Of course, the latter notions can be proven to be equivalent. The proof is surprisingly similar to that of the group case (see Theorem 2.1.5). For a proof of the following see [21, Theorems 25, 32] and [8, Theorem 4.6].

**Theorem 2.2.8.** *Let $(X, d)$ be an extended metric space of bounded geometry. Then the following conditions are equivalent:*

*(1) $(X, d)$ has an invariant measure.*

*(2)* $(X, d)$ *satisfies the Følner condition.*

*(3)* $(X, d)$ *is not paradoxical.*

*(4) The algebra* $C_{u,alg}(X, d)$ *is algebraically amenable (see Section 2.3.3).*

*(5) The uniform Roe algebra* $C_u^*(X, d)$ *admits a, necessarily amenable, trace.*

*(6) The uniform Roe algebra* $C_u^*(X, d)$ *is not properly infinite.*

*(7)* $[0] \neq [1]$ *in the* $K_0$*-group of* $C_u^*(X, d)$*.*

We, therefore, may say that an extended metric space is *amenable* when it satisfies any of the equivalent conditions in Theorem 2.2.8. Using the Følner-type characterization (2) above one can then prove that the amenability of a space is a quasi-isometric invariant (see Theorem 5.3.5 for a similar proof in the context of inverse semigroups).

**Proposition 2.2.9.** *Let* $(X, d_X)$ *and* $(Y, d_Y)$ *be two quasi-isometric extended metric spaces of bounded geometry. If* $Y$ *is amenable, then so is* $X$*.*

As with the case for groups and algebras, one might be interested in the difference between the proper Følner condition and the usual Følner condition. Extended metric spaces, in that regard, behave similarly to algebras. In particular, see Lemmas 2.1.3 and 2.1.9 and compare them with the next Example 2.2.10 and Lemma 2.2.11.

**Example 2.2.10.** Let $(N, d_N)$ be a non-amenable metric space, such as the Cayley graph of a non-abelian free group. Consider the extended metric space $X := N \sqcup \{p\}$, where $p \notin N$, equipped with the extended metric $d(p, N) = \infty$ and $d|_N = d_N$. It then follows that $X$ is an amenable metric space, with Følner set given by $\{p\}$ or invariant measure given by $\mu = \delta_p$. However, it does not satisfy the proper Følner condition, because, if it did, then $N$ would amenable. The next proposition in fact shows that this is the only possible case, for a proof see [7, Corollary 2.20].

**Lemma 2.2.11.** *Let* $(X, d)$ *be an extended metric space. Suppose* $X$ *is Følner but not properly Følner. Then* $X$ *decomposes as* $X = X_g \sqcup X_b$*, where* $X_g$ *is finite,* $X_b$ *is non-amenable and* $X_g$ *and* $X_b$ *are at infinite distance of each other.*

*Property A* is a metric property of a space that can be seen as a weak version of amenability (cf., [76, Proposition 4.1.4]). This point of view, however, is not convenient for this thesis. Indeed, Yu introduced property A in [119] as a non-equivariant amenability and, therefore, property A *is* a weak version of amenability. However, this is no longer true for the metric space scenario, where the relation between these two notions is more intricate, for there are non-amenable property A spaces and non-property A amenable spaces, see Section 5.4.5. See, for example, [89, 76, 113] and references therein for a comprehensive introduction to the topic.

**Definition 2.2.12.** Let $(X, d)$ be a metric space of bounded geometry. Likewise, let $\{(X_e, d_e)\}_{e \in E}$ be a family of metric spaces of bounded geometry.

(1) $X$ has *property A* if for every $r, \varepsilon > 0$ there is $c > 0$ and $\xi\colon X \to \ell^1(X)$ such that:

   (a) For every $x \in X$ the function $\xi_x$ is positive, has norm 1 and its support is contained in $B_c(x)$, the ball of center $x$ and radius $c$.
   (b) $\|\xi_x - \xi_y\|_1 \leq \varepsilon$ for every $x, y \in X$ such that $d(x, y) < r$.

(2) The family $\{(X_e, d_e)\}_{e\in E}$ has *uniform property A* if every $(X_e, d_e)$ has property A with uniformly bounded constants $c_e > 0$, i.e., $\sup_{e\in E} c_e < \infty$.

It is not customary to do so, but we introduced uniform property A in Definition 2.2.12 since it will be useful for us when considering property A for inverse semigroups. In that scenario the metric space decomposes into its connected components, and each one comes naturally equipped with a label $e \in E$, for a certain set $E$. Moreover, we want to remark that the existence of both $\xi$ and $c$ is essential. Indeed, one might think that the constant is not crucial but, as far as the author knows, every construction of a non-A space actually constructs a sequence of spaces $\{X_e\}_{e\in E}$ whose coarse disjoint union is not A in the sense that the constants $c_e$ are arbitrarily large. For instance, a coarse disjoint union of graphs of large girth and bounded degree does not have property A (see [114] or [76, Theorem 4.4.9]). In the same vein see [75] and [76, Theorem 4.5.3] for a proof that the coarse disjoint union of $\{G^n\}_{n\in\mathbb{N}}$, where $G$ is a finite group of order at least 2, does not have property A.

Historically, property A was introduced as a sufficient condition to ensure that $(X, d)$ coarsely embeds into a Hilbert space (see [76, 89, 119] and the survey [113]). For a proof of the following see, for instance, [76, Theorem 5.2.4].

**Proposition 2.2.13.** *Any uniformly discrete metric space with property A coarsely embeds into a Hilbert space.*

The latter sufficient condition is a particularly interesting notion in the context of the so-called *Baum-Connes conjecture* since it is well known (see [119]) that any group that coarsely embeds into a Hilbert space satisfies the *Baum-Connes conjecture with coefficients.* To conclude this utterly brief and very incomplete introduction to property A we recall a few elementary facts about property A, the first of which states that it is a coarse invariant. See [119] (or [76, Theorem 4.1.6]) for a proof of the following.

**Proposition 2.2.14.** *Let $X, Y$ be coarsely equivalent metric spaces of bounded geometry. If $Y$ has property A, then so does $X$.*

The last result we state in this section is the following C*-characterization of property A, by means of the uniform Roe algebra of the metric space. Note that the analysis carried out in Section 5.3.2 is heavily influenced by the following result.

**Theorem 2.2.15.** *Let $(X, d)$ be a metric space of bounded geometry. The following assertions are equivalent:*

*(1) $X$ has property A.*

*(2) The uniform Roe algebra $C^*_u(X,d)$ is nuclear.*

*(3) The uniform Roe algebra $C^*_u(X,d)$ is exact.*

*(4) The uniform Roe algebra $C^*_u(X,d)$ is locally reflexive (see [94]).*

*(5) $X$ satisfies the operator norm localization property (see [24]).*

*Proof.* The proof relies on several different results and for this reason we only cite the papers where each implication was first proven. The equivalence between (1) $\Leftrightarrow$ (2) was proven in [100] (see also [20, Theorems 5.1.6 and 5.5.7]). Implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) hold in general. Moreover, Sako proved in [93] that (1) $\Leftrightarrow$ (5) and, then, in [94] that (4) implies (5). $\square$

### 2.2.3 Groups as metric spaces

One of, if not the, most important classes of metric spaces is the one coming from countable groups, which we assume to be discrete and countable throughout the text. Indeed, it is well known (see [89, 76] or Proposition 2.2.17 below) that any such group may be endowed with a *proper* and *right invariant* metric $d$ that is unique up to coarse equivalence.

**Definition 2.2.16.** Let $G$ be a (non-necessarily finitely generated) countable and discrete group. Let $d\colon G\times G\to[0,\infty)$ be a metric.

(1) We say $d$ is *proper* if, for every positive $r>0$ and every $x\in G$, the $r$-ball of $(G,d)$ centered around $x$ is finite.

(2) We say $d$ is *right invariant* if $d(xg,yg)=d(x,y)$ for every $x,y,g\in G$.

**Proposition 2.2.17.** *Let $G$ be a countable and discrete group. Then $G$ has a proper and right invariant metric $d$. Moreover, if $d'$ is any other proper and right invariant metric on $G$ then the identity map $id\colon(G,d)\to(G,d')$ is a coarse equivalence.*

*Proof.* From the countability of $G$ one may find finite and symmetric subsets $\Gamma_n\subset G$ such that $\{\Gamma_n\}_{n\in\mathbb{N}}$ is increasing (i.e., $\Gamma_n\subset\Gamma_{n+1}$), is exhausting (i.e., $G=\cup_{n\in\mathbb{N}}\Gamma_n$) and $\Gamma_n\Gamma_m\subset\Gamma_{n+m}$ for every $n,m\in\mathbb{N}$. Then let $\Gamma_0:=\{1\}$ and consider the distance:

$$d\colon G\times G\to[0,\infty), \quad \text{where } d(g,h):=\min\left\{n\in\mathbb{N} \text{ such that } hg^{-1}\in\Gamma_n\right\}.$$

It is routine to show that $d$ is a right invariant metric. Therefore any $r$-ball $B_r(g)$ is isometric to $B_r(1)=\Gamma_r$, which is finite by assumption and, hence, $d$ is proper. For the second statement, given $r\geq 0$ let

$$\rho_-(r):=\min\{d'(g,h) \text{ where } g,h\in G \text{ and } d(g,h)\leq r\}$$
$$\rho_+(r):=\max\{d'(g,h) \text{ where } g,h\in G \text{ and } d(g,h)\leq r\}$$

It is then not hard to prove that $\rho_-, \rho_+$ witness the coarse equivalence of the identity map $id\colon (G,d) \to (G,d')$, that is, $\rho_-(r) \to \infty$ as $r$ grows and for every $g, h \in G$

$$\rho_-\left(d\left(g,h\right)\right) \leq d'\left(g,h\right) \leq \rho_+\left(d\left(g,h\right)\right).$$

□

**Example 2.2.18.** The first example of a proper and right invariant metric comes from considering the case when $G$ is finitely generated, say $G = \langle K \rangle$, where $K = K^{-1}$ is finite. Then one can construct the *(left) Cayley graph of $G$ with respect to $K$*, that is, the graph $\mathrm{Cay}(G,K)$ with vertex set $G$, where $x, y \in G$ are joined by an edge labeled by $k \in K$ if $kx = y$. If $K$ is symmetric, generating and finite then the graph $\mathrm{Cay}(G,K)$ is undirected, connected and $|K|$-regular. Moreover, $d$ is right invariant, since for all $x, y, g \in G$

$$d\left(x,y\right) = d\left(1, yx^{-1}\right) = d\left(1, yg\left(xg\right)^{-1}\right) = d\left(xg, yg\right).$$

Therefore, by Proposition 2.2.17, the path length metric of the Cayley graph is the unique (up to coarse equivalence) proper and right invariant metric on $G$. In Figure 2.1 and Figure 2.2 we depict several classical examples of Cayley graphs of various groups. Note that, by Proposition 2.2.17, the coarse geometry of $G$ does not depend on the choice of the generating set $K$ and, therefore, the two depictions of $\mathbb{Z}$ in Figure 2.1 are coarsely equivalent (and even quasi-isometric by Proposition 2.2.5 or by *common sense*).

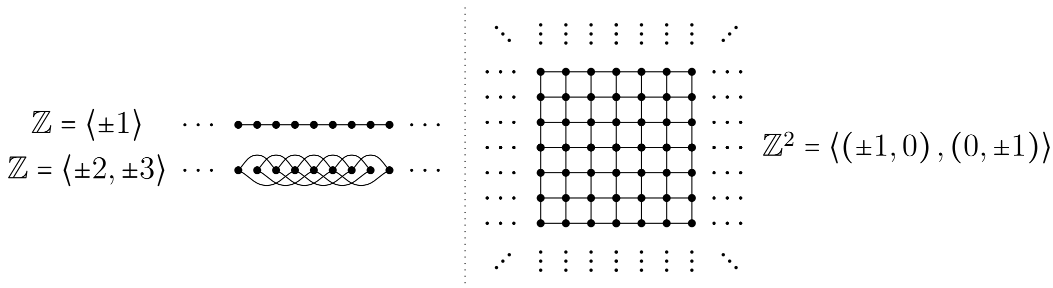


FIGURE 2.1: Cayley graphs of $\mathbb{Z}$ with respect to the generating sets $\{\pm 1\}$ and $\{\pm 2, \pm 3\}$ and $\mathbb{Z}^2$ with respect to the generating set $\{(\pm 1, 0), (0, \pm 1)\}$.

Having equipped any countable and discrete group with a proper and right invariant metric we go onto studying the two coarse invariant properties mentioned earlier, namely *amenability* (see Definition 2.1.1) and *property A* (see Definition 2.2.12). In particular, one may see a group $G$ as an algebraic object and study its amenability, or one may see it as a large-scale geometry object and study its metric amenability. The following, whose proof is routine, states that these two points of view coincide.

**Proposition 2.2.19.** *Let $G$ be a countable and discrete group. Then $G$ is amenable as a group if and only if it is amenable as a metric space (when equipped with its unique proper and right invariant metric).*

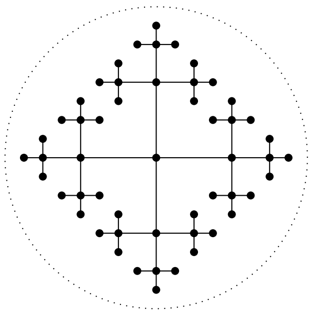


FIGURE 2.2: Cayley graph of the non-abelian free group on two generators $\mathbb{F}_2$ with respect to the usual generating set $\{a^{\pm 1}, b^{\pm 1}\}$.

The rest of the section aims to introduce property A in the case of groups (and their related metric spaces). First of all, note that property A groups exist in abundance:

**Proposition 2.2.20.** *Both amenable groups and free groups have property A.*

*Proof.* We only sketch the proofs.

Given $r, \varepsilon > 0$, the set $\mathcal{F} \subset G$ of elements of $G$ of length not greater than $r$ is finite. By the Følner characterization of amenable groups (see Theorem 2.1.5 (2)) there is a finite $(\mathcal{F}, \varepsilon)$-invariant subset $F \subset G$ containing the identity $1 \in G$. Consider then the uniform measure given by $\zeta_x := 1_{Fx}/|F|$, where $1_{Fx}$ denotes the characteristic function of $Fx \subset G$. It is clear that $\zeta_x$ is a positive, norm 1 element of $\ell^1 G$. Moreover $\operatorname{supp}(\zeta_x) \subset Fx$ which, in turn, is contained in a finite ball around $x \in G$. Finally, following the left invariance of the Følner set $F$, one may prove that $\|\zeta_x - \zeta_y\| \leq \varepsilon$ whenever $x, y \in G$ are closer than $r$.

The proof that free groups (and, more generally, trees) have property A is contained in [76, Example 4.1.5], but the idea is simple enough to describe it. Let $T$ be an infinite tree. Choose an arbitrary vertex $v \in T$ and an infinite geodesic ray $\omega$ starting at $v$. For any other vertex $x \in T$ there is a unique geodesic ray $\omega_x$ starting at $x$ and such that $\omega \cap \omega_x$ is infinite. Thus, given $r, \varepsilon > 0$ let $n \leq r/\varepsilon$ and choose $\zeta_x$ to be the normalized characteristic function of the set containing the first $n$ vertices of $\omega_x$. One can then see that $\zeta_x$ witnesses the property A of $T$. □

As was the case for the class of amenable groups, the class of property A groups is preserved under various constructions (see [76, Chapter 4]). Even though it was open for some time whether every countable group had property A it turned out that not every group does. The first proof of the existence of non-property A groups was given by Gromov in [45] and then by Osajda in [78] (see also [113]). These constructions are quite involved, due to the rigid nature of groups, and normally involve *small cancellation properties* and the existence of coarsely embedded *expanders* in the Cayley graph of the group. For the purposes of this text the main interest of property A is the following characterization, proven, mainly, by Ozawa [79] (see also [20, Theorems 5.1.6 and 5.5.7]). Note this is a particular case of Theorem 2.2.15.

**Theorem 2.2.21.** *Let $G$ be a countable and discrete group. The following are equivalent:*

*(1) $G$ has property A.*

*(2) The uniform Roe algebra $C_u^*(G)$ is nuclear.*

*(3) The reduced group C*-algebra $C_r^*(G)$ is exact.*

**Remark 2.2.22.** Observe that the *range* of Theorem 2.2.21 is not the whole class of metric spaces, not even the whole class of undirected and connected graphs. Indeed, not every undirected graph appears as a Cayley graph of some finitely generated group. For instance, the half line $\mathbb{N}$ is not the Cayley graph of any group.

## 2.3 Operator algebras

Yet again the present section aims to introduce another topic vastly used along the thesis, this time C*-algebras. There are numerous reference texts for this but we mostly follow the conventions used by Brown and Ozawa in [20]. Moreover, we shall keep the discussion mostly to the basics. A *C*-algebra $A$* is an involutive Banach algebra that satisfies the so-called *C*-identity*, that is, $\|a\|^2 = \|a^*a\|$ for every $a \in A$. All the maps between algebras we deal with here are considered linear and *-preserving. Moreover, by *representation* of an algebra $A$ we mean a *-homomorphism from $A$ into $\mathcal{B}(\mathcal{H})$, the algebra of bounded linear operators on a complex Hilbert space $\mathcal{H}$. A *trace* on $A$ is a positive linear map $\tau\colon A \to \mathbb{C}$ such that $\tau(ab) = \tau(ba)$, where $a, b \in A$. We say $\tau$ is *normalized at $p$*, where $p \in A$ is an orthogonal projection (i.e., $p = p^2 = p^*$) if $\tau(p) = 1$. Observe that, unless otherwise specified or obviously false, by *trace* we mean *bounded trace*. In addition, see Definition 2.3.7 below for two remarkable classes of traces.

The class of C*-algebras includes *many* interesting examples and constructions. As is well known, a C*-algebra $A$ is commutative if and only if $A = C_0(X)$, that is, $A$ is the algebra of continuous functions on $X$ that vanish at infinity, where $X$ is a locally compact Hausdorff space (the *spectrum of $A$*). Moreover, if $A$ is unital then $X$ is compact (and reciprocally). On the other extreme we have the full matrix algebras $M_n$, where $n \in \mathbb{N}$, which are naturally equipped with the usual tracial state $\mathrm{tr}_n$. In the following, $\mathrm{tr}_n$ denotes the normalized trace of $M_n$, while Tr denotes the non-normalized trace, i.e., $\mathrm{Tr}(1_n) = n$ where $1_n$ is the identity matrix of rank $n$. Observe that Tr extends to an unbounded trace in $\mathcal{B}(\mathcal{H})$. Another algebra of interest is $\mathcal{K}(\mathcal{H})$, the C*-algebra of compact operators on a separable Hilbert space. By the GNS representation (see, e.g., [20]), every C*-algebra can be faithfully represented in some $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ may be assumed separable in case $A$ is separable. Given a *-linear map $\varphi\colon A \to B$ between C*-algebras recall that we say $\varphi$ is

- *contractive* when $\|\varphi(a)\| \le \|a\|$ for every $a \in A$.
- *positive* when it takes the cone of positive elements of $A$ into the cone of positive elements of $B$, that is, $\varphi(a^*a) \ge 0$ for every $a \in A$.

- *completely positive* when every amplification of $\varphi$ is positive, i.e.,

$$\varphi^{(n)}\colon M_n(A) \to M_n(A),$$

defined entry-wise, is a positive map for every $n \in \mathbb{N}$.

As is well known, the representation theory in the context of C*-algebras needs not be as rich as one would like. It is hence customary to study contractive and completely positive maps (c.c.p.) instead of homomorphisms. However, by Stinespring's representation theorem (see [20, Theorem 1.5.3]), every c.c.p. map can be represented as a contraction of a homomorphism between, maybe, *larger* algebras (see [20, Theorem 1.5.3] for details).

Recall that an element $a \in A$ is a *partial isometry* if $a^*a$ is a projection (equivalently, if $aa^*$ is a projection). Finally, recall that, given $n \in \mathbb{N}$, the *Cuntz algebra* $\mathcal{O}_n$ is the universal C*-algebra generated by isometries $\{s_i\}_{i=1}^n$ subject to the relations $s_i s_i^* s_j s_j^* = 0$ whenever $i \neq j$ and $\sum_{i=1}^n s_i s_i^* = 1$. Observe that $\mathcal{O}_2$, for instance, can be represented in $\mathcal{B}(\ell^2\mathbb{N})$ via the operators

$$s_1 e_n \coloneqq e_{2n} \quad \text{and} \quad s_2 e_n \coloneqq e_{2n+1},$$

where $\{e_n\}_{n\in\mathbb{N}}$ denotes the canonical orthonormal basis of $\ell^2\mathbb{N}$.

The rest of the section introduces several different approximation properties for C*-algebras, namely nuclearity and exactness in Section 2.3.1; and hypertraciality and quasi-diagonality in Section 2.3.2. Lastly, in Section 2.3.3 we give a survey of known results between these approximation properties and some other approximation properties for certain particular constructions, namely *group C*-algebras* and *uniform Roe algebras*.

### 2.3.1 Nuclearity and exactness

*Nuclearity* and *exactness* of a C*-algebra have been extensively studied and, therefore, we do not plan on covering every topic here. We just recall some known (albeit not basic) facts about exact and nuclear algebras. The proof of most of the following results can be found in [20, Chapter 2].

**Definition 2.3.1.** Let $A, B$ be separable C*-algebras.

(1) A homomorphism $\phi\colon A \to B$ is *nuclear* if, and only if, it approximately factors through finite-dimensional algebras, that is, there is a sequence of positive numbers $k(n) \in \mathbb{N}$ and c.c.p. maps $\varphi_n\colon A \to M_{k(n)}$ and $\psi_n\colon M_{k(n)} \to B$ such that for every $a \in A$

$$\|\phi(a) - \psi_n \circ \varphi_n(a)\| \xrightarrow{n\to\infty} 0.$$

(2) The C*-algebra $A$ is *nuclear* if the identity map $id\colon A \to A$ is nuclear.

(3) The C*-algebra $A$ is *exact* when any of its faithful representations is nuclear (as an inclusion map into $\mathcal{B}(\mathcal{H})$).

Observe that any nuclear C*-algebra is also exact. Moreover, while exactness passes to sub-C*-algebras in general, nuclearity does not. Indeed, the *uniform Roe algebra* (see Section 2.3.3) of the free group $\mathbb{F}_2$ is nuclear (see Theorem 2.2.15), while the *reduced C*-algebra* of $\mathbb{F}_2$ is not (see Theorem 2.1.5). In general, the difference between exactness and nuclearity is the target of the maps $\psi_n$ in Definition 2.3.1 above. Indeed, for arbitrary exact algebras one cannot force the image of $\psi_n$ to be contained in $A$.

The class of nuclear C*-algebras encompasses many interesting examples. For instance, one may use a partition of unity argument to prove that every abelian algebra is nuclear. More generally, every *subhomogeneous* C*-algebra is nuclear:

**Proposition 2.3.2.** *Let $q > 0$ and let $X$ be a locally compact Hausdorff space. Every C*-subalgebra $A \subset C_0(X) \otimes M_q$ is nuclear.*

*Proof.* Observe that every irreducible representation of $A$ is of dimension, at most, $q$ and, thus, $A$ is *subhomogeneous* (see [20, Definition 2.7.6]). Hence $A$ is nuclear by [20, Proposition 2.7.7]. □

Moreover, in order to check the nuclearity of a certain C*-algebra one needs only check whether the identity map factors through another nuclear algebra, not necessarily a finite dimensional full matrix algebra.

**Proposition 2.3.3.** *A C*-algebra $R$ is nuclear if and only if for every finite $\mathcal{F} \subset R$ and $\varepsilon > 0$ there is a nuclear C*-algebra $A$ and completely positive and contractive maps $\varphi: R \to A$ and $\psi: A \to R$ such that $\|\psi \circ \varphi(a) - a\| \leq \varepsilon$ for every $a \in \mathcal{F}$.*

*Proof.* The claim follows applying the nuclearity of $A$ and an $\varepsilon/2$-argument. □

The following is a deep theorem, proven by Kirchberg and Phillips (see [54, Theorem 2.8]), and then used multiples times in the literature. Note that, since exactness passes to subalgebras, one direction below amounts to proving that $\mathcal{O}_2$ is exact, which is well known.

**Theorem 2.3.4.** *A separable C*-algebra $A$ is exact if and only if it embeds into $\mathcal{O}_2$.*

Exact algebras, however, were not initially introduced as in Definition 2.3.1. Indeed, the following well known fact was initially proven by Kirchberg (for a proof see [20, Theorem 3.9.1]).

**Theorem 2.3.5.** *A C*-algebra $A$ is exact if and only if for every C*-algebra $B$ and every ideal $J \lhd B$ the sequence $0 \to J \otimes A \to B \otimes A \to (B/J) \otimes A \to 0$ is exact.*

In the same vein, nuclear algebras may be characterized as the ones whose algebraic tensor product with any other C*-algebra has a unique C*-norm. Note that the following result again immediately implies that nuclear algebras are exact, since the maximal tensor product preserves exact sequences. As usual, for a proof see [20, Theorem 3.8.7].

**Theorem 2.3.6.** *A C*-algebra $A$ is nuclear if and only if the algebraic tensor product $A \odot B$ has a unique C*-norm for any C*-algebra $B$.*

### 2.3.2 Traces and Følner-type approximations

Amenability for groups (see Section 2.1 and, in particular, Theorem 2.1.5) has many generalizations to the context of C*-algebras. Many of these notions, such as *nuclearity* or *quasi-diagonality*, turn out to no longer be related in the C*-scenario, while others, such as *hypertraciality* and proper infiniteness, do keep a close relation. This section aims to provide all the necessary definitions, and state some important results about these notions.

The study of amenability in operator algebras[3] was first considered by Connes in his seminal work [26], where he essentially translated the group-theoretic Følner condition into the operator algebra case, see Definition 2.3.7 (2), and related it with the existence of a certain *invariant* trace, see Definition 2.3.7 (1) below. This relation was then made explicit by Bédos in [14, 15], where he also studied applications of these approximations to the study of the spectrum of the positive operators within the algebra (see, e.g., [14, Theorem 1.3]). Item (3) in the definition below goes back to the original work of Murray-von Neumann on, for instance, type III algebras. For further references and motivations see [6, 8, 9, 20] and references therein.

**Definition 2.3.7.** Let $A \subset \mathcal{B}(\mathcal{H})$ be a unital C*-algebra of bounded linear operators on a complex separable Hilbert space $\mathcal{H}$. A state is a positive and linear functional on $A$ with norm one (it is, in particular, normalized).

(1) A state $\tau$ on $A$ is called an *amenable trace* if there is a state $\phi$ on $\mathcal{B}(\mathcal{H})$ extending $\tau$, i.e., $\phi|_A = \tau$, and satisfying

$$\phi(at) = \phi(ta)$$

for every $a \in A$ and $t \in \mathcal{B}(\mathcal{H})$. In this case, the state $\phi$ is called a *hypertrace*. A C*-algebra $A$ is a called *hypertracial* if it has a hypertrace.[4]

(2) $A$ satisfies the *Følner condition* if for every $\varepsilon > 0$ and every finite $\mathcal{F} \subset A$ there is a non-zero finite rank orthogonal projection $p \in \mathcal{B}(\mathcal{H})$ such that $\|pa - ap\|_2 \leq \varepsilon \|p\|_2$ for every $a \in \mathcal{F}$, where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm.

(3) A projection $p \in A$ is *properly infinite* if there are $v, w \in A$ such that $p = v^*v = w^*w \geq vv^* + ww^*$. Note that, in this case, the range projections $vv^*$ and $ww^*$ are necessarily orthogonal. The algebra $A$ is called *properly infinite* when its identity $1 \in A$ is properly infinite.

[3] Connes studies von Neumann algebras in [26], but his computations can also be carried out in the C*-case.

[4] Hypertracial C*-algebras are also called Følner C*-algebras within the literature, see for example [6, 8]. However, the difference between hypertrace are amenable trace are customary (see [20, Section 6.2], for instance).

We first wish to establish the relation between the latter notions since, as in the group case, they are closely related.

**Theorem 2.3.8.** *Let $A$ be a unital separable C*-algebra. Then the following are equivalent:*

*(1) There is a faithful representation $\pi: A \to \mathcal{B}(\mathcal{H})$ such that $\pi(A)$ satisfies the Følner condition.*

*(2) There is a non-zero representation $\pi: A \to \mathcal{B}(\mathcal{H})$ such that $\pi(A)$ has an amenable trace.*

*(3) For every $\varepsilon > 0$ and every finite $\mathcal{F} \subset A$ there is a unital completely positive map $\varphi: A \to M_k$, where $k > 0$, such that for all $a, b \in \mathcal{F}$*

$$\|\varphi(ab) - \varphi(a)\varphi(b)\|_{2,tr} \leq \varepsilon$$

*where $\|m\|_{2,tr} := tr((m^*m)^{1/2})$.*

*Proof.* The proof virtually boils down to the work of Connes [26]. For a more recent proof see, for instance, [20, Theorem 6.2.7] and [6, Theorem 4.3]. □

The main relation we want to point out between hypertraciality and nuclearity is the following well known fact. For a proof of the following see, e.g., [20, Proposition 6.3.4].

**Proposition 2.3.9.** *Every trace on a unital and nuclear C*-algebra is amenable.*

Observe that one of the most important results in the study of C*-algebras of the last decade, namely the Tikuisis-White-Winter theorem (see Theorem 2.3.11), is actually a strengthening (modulo the UCT assumption) of the above result. Indeed, note that *quasidiagonal* traces (see Definition 2.3.10) are, in particular, amenable. However, obtaining a 2-norm approximation is, in general, much easier than obtaining a norm approximation. On the one hand, for the former one may resort to the work of Connes and deduce the existence of such Følner sequence from the uniqueness of the hyperfinite $\mathrm{II}_1$ factor. Meanwhile, the latter requires a much more delicate study, be it in terms of *spectral* manipulation (as in [77, 107, 40]) or in terms of the study of the associated KK-groups (as in [96]).

*Quasidiagonality* was introduced by Halmos in [47] for a single operator. He introduced it in relation with the invariant subspace problem in operator theory, but it has found great many uses in operator algebras. From [47, Section 4], we say $t \in \mathcal{B}(\mathcal{H})$ is *quasidiagonal* if and only if there is a sequence $\{p_n\}_{n\in\mathbb{N}}$ of finite-rank orthogonal projections strongly converging to the identity operator and asymptotically commuting with $t$.

Using the above notion of quasidiagonal operator one can say that a concretely defined C*-algebra $A \subset \mathcal{B}(\mathcal{H})$ is quasidiagonal if all its elements share such an asymptotically commuting sequence of projections $\{p_n\}_{n\in\mathbb{N}}$ (see, e.g., [28, Chapter IX.8]). This definition has the handicap of being, a priori, dependent on the representation of the algebra. One can overcome this by using what is nowadays the customary definition of a quasidiagonal algebra, due to Voiculescu [111, 112] (see also [20, Definition 7.7.1]).

**Definition 2.3.10.** Let $A$ be a separable C*-algebra and $\tau$ a trace on $A$.

(1) We say $A$ is *quasidiagonal* if for every $\varepsilon > 0$ and finite $\mathcal{F} \subset A$ there is some $k > 0$ and completely positive and contractive map $\varphi: A \to M_k$ such that $\|\varphi(a)\| \geq \|a\| - \varepsilon$ and $\|\varphi(ab) - \varphi(a)\varphi(b)\| \leq \varepsilon$ for all $a, b \in \mathcal{F}$.

(2) We say $\tau$ is *quasidiagonal* if for every $\varepsilon > 0$ and finite $\mathcal{F} \subset A$ there is some c.c.p. map $\varphi: A \to M_k$ as in (1) such that $|\tau(a) - \mathrm{tr}_k(\varphi(a))| \leq \varepsilon$ for every $a \in \mathcal{F}$.

As has been mentioned in the discussion following Proposition 2.3.9, quasidiagonal traces are always amenable, since the 2-norm approximation appearing in Definition 2.3.7 is weaker than the norm approximation appearing in Definition 2.3.10 above. However, it turns out that one may obtain a norm approximation when the algebra is nuclear and has a faithful trace. The following is a deep and important result, whose proof can be found in [107, Theorem A], along with the precise definition of *UCT*.

**Theorem 2.3.11.** *Let $A$ be a separable nuclear C*-algebra satisfying the UCT. Then any faithful trace on $A$ is quasi-diagonal.*

The Tikuisis-White-Winter theorem above was generalized to exact algebras (instead of nuclear ones) by subsequent work of Gabe (see [40]), where he essentially used the same method of proof to show that it is enough to suppose the algebra is exact and the trace amenable. Moreover, Schafhauser proved the same result in [96], this time using some techniques coming from KK-theory. Note that in all of these proofs the UCT assumption is essential, at least when it comes to the proof itself. Whether the UCT assumption is necessary is an open problem within the literature at the time of the writing of the thesis.

**Theorem 2.3.12.** *Let $A$ be a separable exact C*-algebra satisfying the UCT. Then any faithful amenable trace on $A$ is quasi-diagonal.*

Observe that the amenability assumption in the trace in the theorem above is essential. Indeed, the reduced group algebra $C_r^*(\mathbb{F}_2)$ of the free group on two generators is an exact UCT algebra with a canonical faithful trace. However, that trace cannot be amenable or quasidiagonal, since the group $\mathbb{F}_2$ is not amenable. In particular, the following follows from Theorem 2.3.11, and solved the so-called *Rosenberg's conjecture* (see the appendix of [46]). Furthermore, note that the amenability of the group follows from the quasidiagonality of the algebra by an standard argument. It is the other direction the one that requires a much more delicate study.

**Theorem 2.3.13.** *A discrete group is amenable if and only if its reduced C*-algebra is quasidiagonal.*

### 2.3.3 C*-algebras associated to groups and metric spaces

The present section is dedicated to introduce several classical examples of C*-algebras coming from (countable and discrete) groups and from locally finite extended metric spaces. These C*-algebras have already appeared in results previously stated (such as Theorem 2.1.5 or Theorem 2.2.8), but we define them here.

Given a countable and discrete group $G$, the *group algebra of* $G$ is the set $\mathbb{C}G$ of formal finite sums $\sum_g a_g g$, where $g \in G$ and $a_g \in \mathbb{C}$. Given two such expressions $\sum_g a_g, \sum_h b_h h \in \mathbb{C}G$, their sum is defined in the obvious way, while their product is defined by

$$\left(\sum_{g\in G} a_g g\right)\cdot\left(\sum_{h\in G} b_h h\right) \coloneqq \sum_{g,h} a_g b_h gh = \sum_{s\in G}\left(\sum_{g\in G} a_g b_{g^{-1}s}\right)s,$$

that is, the product in $\mathbb{C}G$ extends linearly the product in $G$. The algebra $\mathbb{C}G$ can then be represented in the Hilbert space $\ell^2 G$ (i.e., the space of square-summable functions from $G$ to $\mathbb{C}$) via the *left regular representation* of $G$, that is, the map $\lambda\colon G \to \mathcal{B}(\ell^2 G)$ defined by $(\lambda_g(f))(h) \coloneqq f(g^{-1}h)$, where $f \in \ell^2 G$ and $g \in G$. It is routine to show that $\lambda_g$ is a unitary operator on $\ell^2 G$ for every $g \in G$. Moreover, letting $\{\delta_x\}_{x\in G} \subset \ell^2 G$ be the canonical orthonormal basis of $\ell^2 G$, one can see that $\lambda_g \delta_x = \delta_{gx}$. The *reduced C*-algebra of* $G$ is the C*-algebra generated by the image of $\lambda$, that is:

$$C^*_r(G) \coloneqq C^*\left(\{\lambda_g\}_{g\in G}\right) \subset \mathcal{B}\left(\ell^2 G\right).$$

Alternatively, one may see $C^*_r(G)$ as the completion of $\mathbb{C}G$ in the norm $\|\cdot\|_s$ given by $\lambda$. However, the norm $\|\cdot\|_s$ needs not be the only C*-norm on $\mathbb{C}G$. Indeed, one may alternatively construct the maximal norm $\|\cdot\|_m$ and close $\mathbb{C}G$ in that norm, yielding the *full C*-algebra of* $G$, denoted by $C^*(G)$. As we noted back in Theorem 2.1.5, the algebras $C^*_r(G)$ and $C^*(G)$ are isomorphic precisely when the group $G$ is amenable.

Following the above regular representation of $G$ one may define a natural action of $G$ on $\ell^\infty(G)$ and form a crossed product $\ell^\infty(G) \rtimes_r G$. Indeed, for any $f \in \ell^\infty(G)$ and $g \in G$ let $(gf)(h) \coloneqq f(g^{-1}h)$, and let

$$\ell^\infty(G) \rtimes_r G \coloneqq C^*\left(\{f\otimes\lambda_g \mid f\in\ell^\infty G, g\in G\}\right) \subset \mathcal{B}\left(\ell^2 G\otimes\ell^2 G\right).$$

Working within $\mathcal{B}(\ell^2 G)$, which is something we will rarely not do, one may alternatively consider the *uniform Roe algebra* $\mathcal{R}_G$ of $G$, which is the C*-algebra generated by $\{\lambda_g\}_{g\in G}$ and $\ell^\infty(G)$ viewed as multiplication operators in $\ell^2 G$, i.e.,

$$\mathcal{R}_G \coloneqq \mathrm{C}^*\Big(\{\lambda_g \mid g\in G\}\cup\ell^\infty(G)\Big) \subset \mathcal{B}\left(\ell^2 G\right).$$

It is straightforward to check that these two algebras are, in fact, isomorphic.

We now recall the construction and some basic properties of the uniform Roe algebra $\mathrm{C}^*_u(X,d)$ of an extended metric space $(X,d)$. We refer to [89, 20, 76, 103, 8, 18] for proofs and additional motivation in the context of metric spaces. Given an extended metric space of bounded geometry $(X,d)$, the *propagation* of a bounded and linear operator $t \in \mathcal{B}(\ell^2 X)$ is defined by

$$p(t) := \sup \left\{ d(x,y) \mid x, y \in X \text{ and } t_{y,x} = \langle \delta_y, t\delta_x \rangle \neq 0 \right\}$$

and $t$ has bounded propagation if $p(t) < \infty$. The *uniform Roe algebra* $\mathrm{C}^*_u(X,d)$ is the C*-algebra generated by the *-algebra $\mathrm{C}^*_{\mathrm{u,alg}}(X,d)$ of operators with bounded propagation. For instance, recall that any countable and discrete group $G$ may be endowed with a proper and right-invariant metric $d$. Using this construction, one may consider the *uniform Roe algebra of* $G$:

$$C^*_u(G) := C^*\left(\left\{t \in \mathcal{B}\left(\ell^2 G\right) \mid p(t) < \infty\right\}\right) \subset \mathcal{B}\left(\ell^2 G\right).$$

We end the section with the following known theorem, which states that the various versions of the uniform Roe algebra of a group coincide. For a proof see, for instance, [20, Proposition 5.1.3].

**Theorem 2.3.14.** *Let $G$ be a countable and discrete group. Then, using the notation above:*

$$\ell^\infty(G) \rtimes_r G \cong \mathcal{R}_G \cong C^*_u(G).$$

The usefulness of the above theorem resides on the fact that each of the descriptions of the algebra might be more amenable than the others in order to carry out certain computations. Note that, for example, using the approximations provided by $\mathcal{R}_G$, where elements can be approximated by linear combinations of terms of the form $f\lambda_g$, is convenient to study the trace space of the algebra (see, e.g., [8]). On the other hand, crossed products are useful to analyze structural aspects of the algebra like, e.g., the relation between the nuclearity of the C*-algebra and the amenability of the action (cf., [20, Theorem 4.3.4]). Finally, the C*-algebra $C^*_u(G)$ captures aspects from the *large-scale geometry* of $G$.

## 2.4 Semigroups

This section, just as the last ones, introduces another topic we use throughout the rest of the thesis, in this particular case semigroup theory. In general, we follow the introductory texts [81, 50, 51, 58]. Other good general references in this area include, but are not limited to, [43, 110, 98, 29, 36, 66].

Recall that a *semigroup* is a non-empty set $S$ equipped with an associative binary operation $(s,t) \mapsto st$. Unless otherwise specified $S$ will be discrete and countable, though not necessarily finitely generated. Moreover, a semigroup is a *monoid* if it has a unit element, usually denoted by $1 \in S$. We say that $S$ has a *zero* element it there is some element $0 \in S$ such that $s \cdot 0 = 0 \cdot S = 0$ for any $s \in S$. It follows that unit and zero elements, if there at all, are unique. We say that an element $e \in S$ is an *idempotent* if $e^2 = e$.

Recall the so-called *Green's relations* (see, for instance, [43]). Given a semigroup $S$, we denote by $S^1$ the monoid $S \sqcup \{1\}$ in case that $S$ is not a monoid already, otherwise $S^1 = S$. Given $s, t \in S$ we say they are:

- $\mathcal{L}$-related, denoted by $s \mathcal{L} t$, when $S^1 s = S^1 t$.
- $\mathcal{R}$-related, denoted by $s \mathcal{R} t$, when $sS^1 = tS^1$.
- $\mathcal{H}$-related, denoted by $s \mathcal{H} t$, when $s \mathcal{L} t$ and $s \mathcal{R} t$.
- $\mathcal{D}$-related, denoted by $s \mathcal{D} t$, when there is some $r \in S$ such that $s \mathcal{L} r$ and $r \mathcal{R} t$.

It follows from the definitions that all $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and $\mathcal{D}$ are equivalence relations on $S$. Moreover, note that $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, while $\mathcal{D}$ is the relation generated by the union of $\mathcal{L}$ and $\mathcal{R}$. As is well known, the $\mathcal{L}$-classes of $S$ correspond exactly to the strongly connected components of the (left) Cayley graph of $S$ (see Section 2.4.1). Moreover, if $L \subset S$ is an $\mathcal{L}$-class and $H \subset L$ is an $\mathcal{H}$-class, then either $H^2 = H$ or $H^2 = \varnothing$ (this is due to Green, see [43]). The first case happens exactly when the $\mathcal{H}$-class contains an idempotent and, in such case, $H$ is a maximal subgroup of $S$. Meanwhile, if $H$ contains no idempotent, then $H^2 = \varnothing$.

Lastly, Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and $\mathcal{D}$ above become particularly simple for inverse semigroups, which is the main case of study in the present thesis (see the end of Section 3.3).

### 2.4.1 (Left) Schützenberger graphs vs. Cayley graphs

Let $S$ be a countable semigroup. We say $S$ *is generated by a set* $K$ when any element $s \in S$ can be written as a finite word $s = k_p \dots k_1$, where each letter $k_i \in K$. In such case we write $S = \langle K \rangle$ and we can form the *(left) Cayley graph of $S$ with respect to* $K$, that is, the graph $\mathrm{Cay}(S, K)$ whose vertex set is given by $S$, and such that two vertices $x, y \in S$ are joined by an edge labeled by $k \in K$ precisely when $kx = y$. The graph $\mathrm{Cay}(S, K)$ is directed and connected. Recall that a *path* connecting $x, y \in S$ is a sequence of vertices $x = x_0, x_1, \dots, x_p = y$, where each pair $(x_i, x_{i+1})$ is an edge in the graph. We say that a subgraph $L \subset \mathrm{Cay}(S, K)$ is a *strongly connected component* if for any pair $x, y \in L$ there is a path connecting $x$ and $y$ and another connecting $y$ and $x$ (note these paths may differ, even in length). We say the *Schützenberger graphs of $S$ (with respect to $K$)* are the strongly connected components of the (left) Cayley graph of $S$ with respect to $K$. It turns out that Schützenberger graphs of $S$ exactly correspond to the $\mathcal{L}$-classes of the semigroup:

**Proposition 2.4.1.** *Let $S = \langle K \rangle$ be a semigroup, and let $L \subset S$. Then $L$ is an $\mathcal{L}$-class if and only if $L$ is a Schützenberger graph of $S$ with respect to $K$.*

The above proposition allows us to use the terminology $\mathcal{L}$*-class*, *strongly connected component*, *component* or *Schützenberger graph* indistinctly, which we shall do throughout the rest the text. This, even if confusing, is advantageous. Indeed, we shall say $\mathcal{L}$-class when we wish to study it algebraically, while we will say *Schützenberger graph* if we want to equip it with the path distance and the labeling of the edges. See Figure 2.3 for some (left) Cayley graphs.

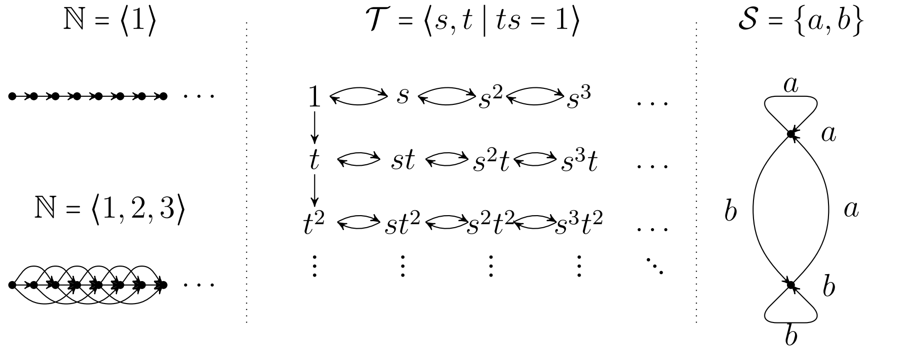


FIGURE 2.3: Cayley graphs of: $\mathbb{N}$ with respect to $\{1\}$ and $\{1,2,3\}$; $\mathcal{T} = \langle s,t \mid ts = 1\rangle$ with respect to $\{t,s\}$; and $\mathcal{S} = \{a,b\}$, where $ab = aa = a$ and $ba = bb = b$, with respect to $\{a,b\}$.

### 2.4.2 Day's amenability in general semigroups

The notions of amenability and Følner sequences do have an analogue in the semigroup scenario, though classical equivalences as in Theorem 2.1.5 do not hold in general, as we shall see. Firstly recall that, given a semigroup $S$, an element $s \in S$ and a set $A \subset S$, the *preimage of $A$ by $s$* is defined by

$$s^{-1}A := \{t \in S \mid st \in A\},$$

that is, $s^{-1}A$ is formed by the elements $t \in S$ that land in $A$ when multiplied by $s$ on the left. Note that the preimage of a set $A$ might be drastically different in cardinality than the set itself. Indeed, in the most extreme case $S$ can be infinite and have a zero element $0 \in S$. Then the set $S \setminus \{0\}$ is co-finite, $0^{-1}(S \setminus \{0\})$ is empty, while $0^{-1}\{0\} = S$. This difference in cardinality, byproduct of the non-injectivity of the left multiplication in the semigroup, is troublesome. In the group case, for instance, note that if $g \in G$ and $F \subset G$ then $F$ and $gF$ have the same cardinality and this implies, in particular, that:

$$|gF \setminus F| = |F \setminus gF|. \tag{2.1}$$

The above Eq. (2.1) might seem trivial, but in the semigroup case it does not hold. Indeed, note that, in general, $|sF| \le |F|$, and they actually differ whenever $s$ does not act injectively on $F$. In particular $|sF \setminus F|$ might be small because $|sF|$ is itself small, not because $sF$ and $F$ are similar. Therefore, the Følner condition appearing in Definition 2.1.1 (2) has two possible generalizations to the semigroup case, one that uses $|sF \setminus F|$ and another one that uses $|F \setminus sF|$ (see Definition 2.4.2 (3) and (4)).

By Murphy's law, these two approaches yield different classes of semigroups (see Definition 2.4.2).

In the same fashion, one may look at group amenability and wonder what the *suitable* generalization to semigroups is. On the one hand, one can study finitely additive probability measures that are *image invariant*, meaning that $\mu(sA) = \mu(A)$ for any $s \in S$ and any $A \subset S$. On the other hand, one may study those probability measures that are *preimage invariant*, meaning that $\mu(s^{-1}A) = \mu(A)$. These two notions (see Definition 2.4.2 below and [29, 71, 55, 41]) are not equivalent. For the following, recall that by *probability measure* we mean *finitely additive probability measure*.

**Definition 2.4.2.** Let $S$ be a semigroup.

(1) $S$ is *amenable* if there exists an invariant measure on $S$, i.e., a probability measure $\mu: \mathcal{P}(S) \to [0,1]$ such that for all $s \in S$ and $A \subset S$

$$\mu\left(s^{-1}A\right) = \mu\left(A\right).$$

(2) A semigroup $S$ is called *measurable* if there is a probability measure $\mu$ on $S$ such that for all $s \in S$ and $A \subset S$

$$\mu\left(sA\right) = \mu\left(A\right).$$

(3) $S$ satisfies the *Følner condition* if for all $\varepsilon > 0$ and finite $\mathcal{F} \subset S$ there is a finite non-empty $F \subset S$ such that for every $s \in \mathcal{F}$

$$|sF \setminus F| \leq \varepsilon |F|.$$

(4) $S$ satisfies the *strong Følner condition* if for all $\varepsilon > 0$ and finite $\mathcal{F} \subset S$ there is a finite non-empty $F \subset S$ such that for every $s \in \mathcal{F}$

$$|F \setminus sF| \leq \varepsilon |F|.$$

The amenability definition above is due to Day [29], while the two Følner notions come from Følner's work in groups [39] and the subsequent work of Namioka for semigroups [71]. The measurability condition, as far as the author knows, appeared in Sorenson's Ph.D. thesis [102], and was then studied in Klawe [55]. In the most general case, the following result (due to Namioka, see [71]) gives the only relations between the above that hold for general semigroups.

**Proposition 2.4.3.** *Let $S$ be a countable semigroup. Consider:*

*(1) $S$ is measurable.*

*(2) $S$ satisfies the strong Følner condition.*

*(3) $S$ is amenable.*

*(4) $S$ satisfies the Følner condition.*

*Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Moreover, none of the reverse implications hold.*

**Remark 2.4.4.** Apart from the notions of amenability/Følner's condition stated above, there are still other notions that have appeared in the literature. Among these, a remarkable one is *fair amenability*, due to Deprez in [30] (see also Remark 4.1.7 below for a brief discussion about it).

We end the section and chapter with a hand-waving digression about the suitability of working in the generality of arbitrary semigroups (this shall be a recurring theme in the next Chapter 3). Indeed, we claim that, within this generality, the amenability notion as defined by Day in [29] (see Definition 2.4.2 (1)) does not meet some requirements we might be used to when talking about *amenable structures*[5]. Consider, for instance, the two-element semigroup $\mathcal{S} = \{a, b\}$, where $aa = ab = a$ and $bb = ba = b$ (see Figure 2.3 for a depiction of its left Cayley graph). The semigroup $\mathcal{S}$ satisfies the Følner condition, since it is finite and we may take $F := \mathcal{S}$, but it is not amenable since any invariant measure $\mu$ must satisfy $\mu(b^{-1}\{a\}) = \mu(a^{-1}\{b\}) = \mu(\varnothing) = 0$, and hence

$$\mu(\mathcal{S}) = \mu(\{a\}) + \mu(\{b\}) = \mu\left(b^{-1}\{a\}\right) + \mu\left(a^{-1}\{b\}\right) = 0 + 0 = 0.$$

Therefore no probability measure on $\mathcal{S}$ can be invariant and, thus, $\mathcal{S}$ is not amenable. Note that $\mathcal{S}$ does not satisfy the strong Følner condition either, since letting $\mathcal{F} = \mathcal{S}$ the whole semigroup $F := \mathcal{S}$ cannot be strongly Følner, for $|F \setminus aF| = |F \setminus bF| = 1$, while neither can $F' := \{a\}$ or $F'' := \{b\}$.

From a different point of view, one may wonder how the representation theory (in Hilbert spaces) of a semigroup looks like. In particular, a natural question is whether there is a *left regular representation* of a semigroup $S$. Following a naive approach one constructs the Hilbert space $\ell^2 S$, and then *represents* $S$ in $\ell^2 S$ in the usual group-theoretic way:

$$\pi\colon S \to \mathcal{B}\left(\ell^2 S\right), \ \pi_s \delta_t := \delta_{st}, \tag{2.2}$$

where $\{\delta_t\}_{t\in S}$ is the canonical orthonormal basis of $\ell^2 S$. The above Eq. (2.2) is, of course, problematic. For instance, if $S$ is an infinite semigroup with a zero element, then $\pi_0$ is not a bounded operator in $\ell^2 S$. The problem arises since the left multiplication by 0 shrinks an infinite set into a finite one, and this kind of behavior is common within semigroup theory. However, in a sense, left multiplication by 0 is not problematic should one restrict to some kind of *domain within* $S$, i.e., a set $D_0 \subset S$ such that 0 acting on $D_0$ does not shrink the size of any infinite set into a finite one. In this case, for instance, it would be sufficient to consider any finite set $D_0 \subset S$. Should one be able to construct such a *domain* $D_s$ for any element in $S$,

[5] In whichever category we may choose to work.

then the following

$$\pi\colon S \to \mathcal{B}\left(\ell^2 S\right), \; \pi_s\delta_t := \begin{cases} \delta_{st} & \text{if } t \in D_s \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

might yield a *natural left regular representation* of $S$. Of course, in order for Eq. (2.3) to be a semigroup homomorphism one would need some compatibility conditions among the collection of domains, but we are not going to delve into that. There is, however, no natural choice for the collection $\{D_s\}_{s\in S}$ unless in some particular classes of semigroups, such as *cancellative* semigroups and *inverse* semigroups. Note that this last class shall be the main object of study of the actual contents of this thesis. In the most general case, one can follow a routine Zorn's Lemma argument to prove that there indeed exists a maximal family $\{D_s\}_{s\in S}$ of domains $D_s \subset S$ such that left multiplication by $s$ on $D_s$ defines an injective map. This, however, serves no purpose whatsoever, as the domains appearing may, in general, have no inherent algebraic properties apart from the injectivity condition.

All in all, left multiplication in arbitrary semigroups is too wild to have some *correct* notion of amenability or representation. This, as we mentioned, will be a recurring theme within the next chapter.

# Chapter 3
# Semigroups and algebraic amenability

Properly speaking the thesis starts here. In this regard, and contrary to Chapter 2, most results from now on are *new* and have been developed for the present thesis. For this reason every *non-new* result we state will be properly indicated and cited.

The present chapter aims to begin the study of amenability (in Day's sense, see Definition 2.4.2) of semigroups. It shall be brief, and we mainly focus of the fact that there cannot be a satisfying dichotomy between *amenability* and *paradoxicality* (see Section 3.1) within this generality. This chapter is, therefore, a *non-chapter*, meaning we will indicate why such a dichotomy cannot be achieved. This fact will oblige us to restrict ourselves to some *nicer* category, namely cancellative semigroups, and then inverse semigroups where, as will be seen in Chapter 4, the classical dichotomy can be proven in a satisfactory manner. Therefore, and from a *narrative* point of view, this chapter's reason to exist is to justify the setting we consider in the upcoming Chapter 4. We will also introduce here some natural C*-algebras associated to these special classes of semigroups.

The following proposition is readily seen and justifies why we can assume a semigroup $S$ to be countable and unital, as we will normally do in the upcoming sections. In general, given a (possibly non-unital) semigroup $S$ we can always consider its unitization $S' := S \sqcup \{1\}$ and define a multiplication in $S'$ extending that of $S$ so that 1 behaves as a unit. Moreover, as in the case of groups and algebras, the property of amenability is in essence a countable one, at least for a large class of semigroups (including the inverse). Recall from [42] that a semigroup $S$ satisfies the *Klawe condition* whenever $sx = sy$ for $s, x, y \in S$ implies there is some $t \in S$ such that $xt = yt$. As mentioned in [42], the Klawe condition is very general and, in particular, left cancellative as well as inverse semigroups satisfy it.

**Proposition 3.0.1.** *Let $S$ be a semigroup and denote by $S'$ its unitization. Then*

*(i) $S$ is amenable if and only if $S'$ is amenable.*

*(ii) If any countable subset in $S$ is contained in an amenable countable subsemigroup of $S$, then $S$ is amenable. If, in addition, $S$ satisfies the Klawe condition, then the reverse implication is also true.*

*Proof.* (i) The proof directly follows from the definition. Indeed, an invariant measure on $S$ can be extended to an invariant measure on $S'$ defining $\mu(\{1\}) = 0$. Conversely, $\{1\}$ is a null set for any invariant measure on $S'$, so any invariant measure on $S'$ is also an invariant measure on $S$.

(ii) For the first part, let $\mathcal{A}(S)$ denote the set of countable and amenable subsemigroups of $S$. Furthermore, for $T \in \mathcal{A}(S)$ denote by $\mu_T$ an invariant measure on $T \subset S$. We may, without loss of generality, extend it to $S$ by defining $\mu_T(S \setminus T) := 0$. Observe that then $\mu_T(t^{-1}A) = \mu_T(A)$ for every $t \in T$, $A \subset T$. Consider the measure:

$$\mu\colon \mathcal{P}(S) \to [0,1], \quad A \mapsto \mu(A) := \lim_{T \in \mathcal{A}(S)} \mu_T(A) = \lim_{T \in \mathcal{A}(S)} \mu_T(A \cap T),$$

where the limit is taken along a free ultrafilter of $\mathcal{A}(S)$. It follows from a straightforward computation that $\mu$ is an invariant measure on $S$.

For the second part we follow a similar route to that of [7, Proposition 3.4]. Recall from [42] that a semigroup satisfying the Klawe condition is amenable if and only if it satisfies the strong Følner condition, that is, for every $\varepsilon > 0$ and finite $\mathcal{F} \subset S$ there is a $(\varepsilon, \mathcal{F})$-Følner set $F \subset S$ such that $|F| = |sF|$ for every $s \in \mathcal{F}$ (see Theorem 2.6 in [42]).

Let $C = \{c_n\}_{n\in\mathbb{N}} \subset S$ be a countable subset. In order to construct an amenable semigroup $T \supset C$ we define an increasing sequence $\{T_n\}_{n\in\mathbb{N}}$ of countable subsemigroups of $S$ by:

- $T_0$ is the subsemigroup generated by $C$.
- Suppose $T_i = \{t_j\}_{j\in\mathbb{N}}$ has been defined. By the previous paragraph for every $k \in \mathbb{N}$ we may find an $(1/k, \{t_1, \dots, t_k\})$-Følner set $F_k \subset S$ such that $|t_j F_k| = |F_k|$ for every $j = 1, \dots, k$. We thus define the semigroup $T_{i+1}$ to be the semigroup generated by $T_i \cup F_1 \cup F_2 \cup \dots$.

Then $T = \cup_{i\in\mathbb{N}} T_i$ is amenable, countable and contains $C$. □

As in the case of metric spaces or algebras (see, e.g., [8, Section 2.1] and [6, Section 4]), an amenable semigroup can have non-amenable sub-semigroups. For instance, take $S := \mathbb{F}_2 \sqcup \{0\}$, where $0\omega = \omega 0 = 0$ for every $\omega \in \mathbb{F}_2$. This semigroup is amenable, since it has a zero element, but has a non-amenable sub-semigroup.

## 3.1 Classical equivalences

By *classical equivalences* we mean the relation between Følner condition, amenability and non-paradoxicality. In the first result relating these notions we establish the difference between the Følner condition and the proper Følner condition in the context of semigroups. This result is analogous to the behavior in metric spaces (see [8, Proposition 2.15] and [8, Theorem 3.9]). Moreover, its proof is heavily inspired by the corresponding result in the algebra setting [8, Theorem 3.9] (see also Example 2.1.8 and Lemma 2.1.9).

**Theorem 3.1.1.** *Let $S$ be a semigroup. Suppose that $S$ satisfies the Følner condition but not the proper Følner condition. Then there is an element $a \in S$ such that $|Sa| < \infty$.*

*Proof.* Given $\varepsilon > 0$ and a non-empty finite subset $\mathcal{F} \subset S$ define

$$\begin{aligned} \text{Føl}\,(\varepsilon,\mathcal{F}) &:= \left\{ F \subset S \;\middle|\; 0 < |F| < \infty \;\text{ and }\; \max_{s\in\mathcal{F}} \frac{|sF \setminus F|}{|F|} \le \varepsilon \right\}, \\ M_{\varepsilon,\mathcal{F}} &:= \sup_{F\in\text{Føl}(\varepsilon,\mathcal{F})} |F| \in \mathbb{N} \cup \{\infty\}\,. \end{aligned}$$

Since $S$ is *not* properly Følner there is a pair $(\varepsilon_0, \mathcal{F}_0)$ with finite $M_{\varepsilon_0,\mathcal{F}_0}$. Note that the pairs $(\varepsilon,\mathcal{F})$ are partially ordered by $(\varepsilon_1,\mathcal{F}_1) \le (\varepsilon_2,\mathcal{F}_2)$ if and only if $\mathcal{F}_1 \subset \mathcal{F}_2$ and $\varepsilon_2 \le \varepsilon_1$. This partial order induces a partial order on $M_{\varepsilon,\mathcal{F}}$ and thus we may suppose that $\varepsilon_0 M_{\varepsilon_0,\mathcal{F}_0} < 1$. Indeed, simply substitute $\varepsilon_0$ with some $\varepsilon_0' < \min\{\varepsilon_0,\, 1/M_{\varepsilon_0,\mathcal{F}_0}\}$.

We first claim that for any $\varepsilon \in (0,\varepsilon_0]$ and $\mathcal{F} \supset \mathcal{F}_0$ we have $\text{Føl}(0,\mathcal{F}) = \text{Føl}(\varepsilon,\mathcal{F})$. The inclusion $\subset$ is obvious. Moreover, for $F \in \text{Føl}(\varepsilon,\mathcal{F})$ and $s \in \mathcal{F}$ we have

$$|sF \setminus F| \le \varepsilon\,|F| \le \varepsilon M_{\varepsilon,\mathcal{F}} \le \varepsilon_0 M_{\varepsilon_0,\mathcal{F}_0} < 1$$

and hence $|sF \setminus F| = 0$ or, equivalently, $F \in \text{Føl}(0,\mathcal{F})$. Thus it makes sense to consider the largest Følner sets with $\varepsilon = 0$:

$$\text{Føl}_{\max}(0,\mathcal{F}) := \{F \in \text{Føl}\,(0,\mathcal{F}) \;|\; |F| \ge |F'| \text{ for all } F' \in \text{Føl}\,(0,\mathcal{F})\}\,.$$

Next we claim that if $\mathcal{F} \subset \mathcal{F}'$ and $F_m \in \text{Føl}_{\max}(0,\mathcal{F}), F_m' \in \text{Føl}_{\max}(0,\mathcal{F}')$, then $F_m' \subset F_m$. Indeed, suppose the contrary. Then $\widehat{F} := F_m \cup F_m'$ would be in $\text{Føl}_{\max}(0,\mathcal{F})$ and strictly larger than $F_m$, contradicting the maximality condition in the definition of $\text{Føl}_{\max}(0,\mathcal{F})$. In particular, this means that $\text{Føl}_{\max}(0,\mathcal{F})$ has only one element, for if $F_1, F_2 \in \text{Føl}_{\max}(0,\mathcal{F})$ then $F_1 \subset F_2 \subset F_1$.

Finally, denote by $F_{\mathcal{F}}$ the unique element of $\text{Føl}_{\max}(0,\mathcal{F})$ and consider the net $\{|F_{\mathcal{F}}|\}_{\mathcal{F}\in\mathcal{J}}$, where $\mathcal{J} := \{\mathcal{F} \subset S \mid |\mathcal{F}| < \infty \text{ and } \mathcal{F}_0 \subset \mathcal{F}\}$. This net is decreasing and contained in $[1, |F_{\mathcal{F}_0}|] \cap \mathbb{N}$ and, thus, has a limit, which is attained by some $\mathcal{F}_1$. This means that $sF_{\mathcal{F}_1} \subset F_{\mathcal{F}_1}$ for all $s \in S$. Any $a \in F_{\mathcal{F}_1}$ will meet the requirements of the theorem. □

The following known theorem states the only general implications between the Følner condition and amenability (compare with Proposition 2.4.3).

**Theorem 3.1.2.** *Let $S$ be a countable discrete semigroup. Consider the assertions:*

*(1) $S$ is amenable.*

*(2) $S$ has a Følner sequence.*

*(3) $\mathbb{C}S$ is algebraically amenable.*

*Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).*

*Proof.* Følner proved the implication (1) $\Rightarrow$ (2) in the case of groups and, later, Namioka extended the proof for semigroups (cf., [71]). To show (2) $\Rightarrow$ (3) choose a Følner sequence $\{F_n\}_{n\in\mathbb{N}}$ for $S$. Then the linear span of these subsets $W_n :=$

$\mathrm{span}\{f \mid f \in F_n\}$ defines a Følner sequence for $\mathbb{C}S$. In fact, note that $\dim(W_n) = |F_n|$ and for any $s \in S$ we have

$$\frac{\dim(sW_n + W_n)}{\dim(W_n)} \le \frac{|sF_n \cup F_n|}{|F_n|} \xrightarrow{n\to\infty} 1,$$

which concludes the proof. □

We remark that none of the reverse implications in Theorem 3.1.2 hold in general. It is well known that a finite semigroup may be non-amenable and any such semigroup is a counterexample to the implication (2) ⇒ (1), because if $S$ is finite it trivially has a Følner sequence (see $\mathcal{S}$ in Figure 2.3 for a concrete construction).

The following example, although messy, is a counterexample to the implication (3) ⇒ (2) in Theorem 3.1.2.

**Example 3.1.3.** Consider the additive semigroup of natural numbers $\mathbb{N}$ and the free semigroup on two generators $\mathbb{F}_2^+ = \{a, b, ab, \dots\}$, where we assume that the semigroup $\mathbb{F}_2^+$ has no identity. Denote by $\alpha$ the action of $\mathbb{F}_2^+ \curvearrowright \mathbb{N}$ given by $\alpha\colon \mathbb{F}_2^+ \to \mathrm{End}(\mathbb{N})$, $a, b \mapsto \alpha_a(n) = \alpha_b(n) = n - 1$ when $n \ge 1$ and $\alpha_a(0) = \alpha_b(0) = 0$.

We claim the semigroup $S := \mathbb{N} \rtimes_\alpha \mathbb{F}_2^+$ does not satisfy the Følner condition, while its complex group algebra is algebraically amenable. Note that the element

$$s = (0, a) - (1, a) \in \mathbb{C}S$$

clearly satisfies $(n, \omega)s = 0$ for every $(n, \omega) \in S$. Therefore $W := \mathbb{C}s$ is trivially a Følner subspace for $\mathbb{C}S$, since it is a one-dimensional left ideal. This proves that $\mathbb{C}S$ is algebraically amenable.

In order to prove that $S$ does not satisfy the Følner condition we shall prove that for any non-empty finite subset $F \subset S$ either $|(0, a)\, F \smallsetminus F| \ge |F|/50$ or $|(0, b)\, F \smallsetminus F| \ge |F|/50$. First observe that $|(0, a)\, F| \ge |F|/2$, and that equality holds if and only if

$$F = \{(0, w_1), (1, w_1), \dots, (0, w_k), (1, w_k)\} \text{ for some } w_i \in \mathbb{F}_2^+,\ i = 1, \dots, k. \tag{3.1}$$

Indeed, if $F$ is of this form then clearly $|(0, a)\, F| = |F|/2$. And, conversely, given $(n, u) \ne (m, v)$ one has that $(0, a)(n, u) = (0, a)(m, v)$ only when $u = v$ and $n = 0, m = 1$ or $n = 1, m = 0$.

Now suppose $F$ is of the form given in Eq. (3.1) and satisfies $|(0, a)\, F \smallsetminus F| \le |F|/5$. Note that, by the observation in the previous paragraph, $|(0, a)\, F \smallsetminus F| \ge |F|/2 - N_a$, where $N_a$ is the number of words $w_i$ of $F$ that begin with $a$. Since a word cannot begin with $a$ and with $b$, it follows that the number $N_b$ of words that begin with $b$ satisfies $N_b \le |F|/5$. Therefore, again, we conclude that $|(0, b)\, F \smallsetminus F| \ge |F|/2 - |F|/5 \ge |F|/5$, as desired. This proves that no set of the form (3.1) can be Følner.

Finally, given an arbitrary $F \subset S$ we may decompose it into $F = F_* \sqcup F'$, where $F_*$ is of the form (3.1) and $F'$ does not contain pairs of elements of the form

$(0,\omega),(1,\omega)$, with $\omega \in \mathbb{F}_2^+$. We have

$$|((0,a)\,F \cup (0,b)\,F) \smallsetminus F| \geq |(0,a)\,F'| + |(0,a)\,F_*| + |(0,b)\,F'| + |(0,b)\,F_*| - |F| = |F'|\,.$$

Note that the last equality follows from the fact that $|(0,a)\,F'| = |F'| = |(0,b)\,F'|$. Therefore, if $F'$ is relatively large when compared to $F$, then $F$ itself will not be Følner. Suppose hence that $|F'| \leq |F|/25$ and $|(0,a)\,F \smallsetminus F| \leq |F|/25$. Then we have $|F_*| \geq (24/25)\,|F|$. Now observe that

$$(0,a)F_* \smallsetminus F_* = [(0,a)F_* \smallsetminus F] \sqcup [(0,a)F_* \cap F'] \subseteq ((0,a)F \smallsetminus F) \cup F',$$

and so

$$|(0,a)F_* \smallsetminus F_*| \leq \frac{|F|}{25} + \frac{|F|}{25} \leq \frac{2\cdot 25\cdot |F_*|}{25\cdot 24} < \frac{|F_*|}{5}.$$

Since $F_*$ is of the form (3.1), it follows that $|(0,b)\,F_* \smallsetminus F_*| \geq |F_*|/5$. Hence

$$|(0,b)F_* \smallsetminus F_*| \geq \frac{|F_*|}{5} \geq \frac{24|F|}{5\cdot 25}.$$

Finally,

$$\begin{array}{rcl} |(0,b)F \smallsetminus F| & \geq & |(0,b)F_* \smallsetminus F| = |(0,b)F_* \smallsetminus F_*| - |(0,b)F_* \cap F'| \\ & \geq & \dfrac{24|F|}{5\cdot 25} - \dfrac{|F|}{25} \geq \dfrac{|F|}{25}\,. \end{array}$$

It remains to consider what happens when $|F'| \geq |F|/25$. In this case, by the above computation, we get

$$2\max\{|(0,a)F \smallsetminus F|, |(0,b)F \smallsetminus F|\} \geq |((0,a)F \cup (0,b)F) \smallsetminus F| \geq |F'| \geq \frac{|F|}{25},$$

and we deduce that either $|(0,a)F \smallsetminus F|$ or $|(0,b)F \smallsetminus F|$ is greater than or equal to $|F|/50$.

We conclude that no non-empty finite subset $F \subset S$ can be $(\varepsilon,\{a,b\})$-invariant for $\varepsilon < 1/50$, which proves that $S$ itself does not satisfy the Følner condition. □

Looking back at Theorem 3.1.2 one may notice that there is one ingredient from the *classical equivalences* missing, and that is *paradoxicality.* The rest of the section gives a hand-waving explanation of this absence. Observe that one can give a *backwards* definition of *paradoxicality* in the semigroup scenario by saying that a semigroup is *paradoxical* if it has a *paradoxical decomposition via preimages*, that is, there is a partition of $S$ as

$$\begin{aligned} S &= A_1 \sqcup \cdots \sqcup A_n \sqcup B_1 \sqcup \cdots \sqcup B_m \\ &= a_1^{-1}A_1 \sqcup \cdots \sqcup a_n^{-1}A_n = b_1^{-1}B_1 \sqcup \cdots \sqcup b_m^{-1}B_m, \end{aligned}$$

where $a_1, \dots, a_n, b_1, \dots, b_m \in S$. The above definition, although sound, is problematic, for the difference between going *forwards* and *backwards*, on a technical level, poses numerous problems. For instance, one could say two sets $A, B \subset S$ are *equidecomposable* when $A$ can be partitioned into finitely many pieces $A_1, \dots, A_n$ in such a way that $a_1^{-1}A_1, \dots, a_n^{-1}A_n$ forms a partition of $B$, and viceversa. It is then a natural question to wonder whether if $A$ is equidecomposable to a subset of $B$, and $B$ is equidecomposable to a subset of $A$, then $A$ and $B$ are equidecomposable themselves. Note that, for instance, this holds for inverse semigroups, as we will see later (see Lemma 4.2.7 (2)). In the general case, however, the techniques simply do not apply, since they heavily rely on the existence of maps $\phi\colon A \to B$ implementing the equidecomposability of $A$ and $B$. These maps, as the dynamics involved above are taking the preimages of the sets $A_i, B_j$, simply do not necessarily exist.

In the same vein, it is not clear whether the sets $A$ and $a^{-1}A$ are equidecomposable, for the semigroup could *lack elements*. Indeed, take, for instance, $S = (\mathbb{N}, +)$, and $A := \{3\}$. Putting $s = 2$ it is clear that $s^{-1}A = \{1\}$. However, note that $A$ and $s^{-1}A$ are not equidecomposable, for $t^{-1}s^{-1}A = t^{-1}\{1\} = \varnothing$ for all $t \in \mathbb{N}$. In this case the semigroup $S$ *lacks the element* $-2$. This example can be replicated, for instance taking $S := \{a, b\}$, where $aa = ab = a$, $bb = ba = b$, and $A = \{a\}$, as in Figure 2.3.

The essence of the problem arises from the dynamics concerned, that is, arises since Day's semigroup invariance is via taking *preimages* rather than taking *images*. Therefore, in the following, we shall restrict to two classes of semigroups.

## 3.2 Cancellative semigroups

The first example (basic, yet interesting) of class of semigroups for which left multiplication defines an injective map in the semigroup is the class of *left cancellative* semigroups.

**Definition 3.2.1.** A semigroup $S$ is *(left) cancellative* if $s = t$ whenever $rs = rt$ for some $r \in S$.

Since we are dealing only with the *left* variations we drop the *left* prefix henceforth. Groups are clearly cancellative in the sense above, as is any subsemigroup of any group. For instance, the semigroup $\mathbb{N}$ of non-negative integers is left cancellative. In more generality, any concatenation monoid is also left cancellative, i.e., any submonoid of the concatenation monoid $X^*$, where $X$ is a set (known as the *alphabet*), and $X^* = \sqcup_{n \in \mathbb{N}} X^n$, and the product of two strings $\omega, \omega' \in X^*$ is defined to be their concatenation $\omega\omega'$.

The following straightforward proposition shows that these cancellative semigroups have a natural *regular representation.*

**Proposition 3.2.2.** *Let $S$ be a cancellative semigroup. Then $v\colon S \to End(S)$ given by $v_s t := st$ is an injective semigroup homomorphism. In particular, the map $V\colon S \to \mathcal{B}(\ell^2 S)$ given by $V_s\delta_t := \delta_{st}$, where $\{\delta_t\}_{t\in S}$ is the canonical orthonormal basis of $\ell^2 S$, defines a representation of $S$ by isometries.*

*Proof.* The proof is a straightforward computation. The adjoint of $V_s$ is:

$$V_s^* \delta_t = \begin{cases} \delta_r & \text{if } t = sr \\ 0 & \text{otherwise} \end{cases}$$

and, therefore, $V_s^* V_s \delta_r = V_s^* \delta_{sr} = \delta_r$ for every $r \in S$, proving that $V_s$ is an isometry. □

Following the above proposition, one can define the *reduced C*-algebra of S* to be the C*-algebra generated by the image of $V$:

$$C_r^* (S) := C^* (\{V_s \mid s \in S\}) \subset \mathcal{B}\left(\ell^2 S\right).$$

As stated in Proposition 3.2.2, the operators $V_s \in \mathcal{B}(\ell^2 S)$ are isometries, i.e., $V_s^* V_s = 1$. Thus, when taking C*-algebras, one naturally has to consider also inverses of the maps above (denoted by $^*$). Products of these operators $\{V_s, V_s^*\}_{s \in S}$ yield the *inverse hull* of $S$ and, therefore, one naturally gets to *inverse semigroups* (see Section 3.3). In this sense, the elements of the inverse hull of $v(S) \subset \operatorname{End}(S)$ coming from $S$ are exactly the injective maps $v_s \colon S \to S$ given by left multiplication by some element of $S$. Moreover, observe that the bijective maps correspond to the right invertible elements of $S$, i.e., those elements such that $sS = S$.

From the point of view of amenability cancellative semigroups are mostly understood. Indeed, the following was already observed by Namioka in [71, Corollary 4.3] (see also Definition 2.4.2).

**Proposition 3.2.3.** *Let $S$ be a cancellative semigroup. If $S$ satisfies the Følner condition then it is measurable. In particular, all the conditions in Proposition 2.4.3 are equivalent in the class of cancellative semigroups.*

To end the section, we would like to briefly mention that left cancellative semigroups have received a lot of attention lately with regards to C*-algebras. Indeed, and to name only a few, Murphy gave in [69, 70] a construction of the full C*-algebra of a left cancellative semigroup but, as Li points out in [62], this construction is not tailored towards the study of nuclearity, as, for instance, Murphy's full C*-algebra of $\mathbb{N} \times \mathbb{N}$ is not nuclear. This is one of the main motivations in Li's works [62, 63], where he constructs the *full C*-algebra* of a left cancellative semigroup in a manner that resembles previous constructions of Nica [72]. This discussion, however, lies outside of the scope of the thesis, for we will only study reduced C*-algebras.

## 3.3 Inverse semigroups

Following the discussion after Proposition 3.2.2 it seems that the ideal context where one should study amenability for semigroups is that of *inverse semigroups*. These were introduced independently by Wagner (see [109, 110]) and Preston (see [83, 84, 85]) as a generalization of groups. In particular, if elements in a group can be thought of as symmetries of a space, or as bijections of a set, then elements in an inverse semigroup can be thought of as partial symmetries of a space, or as partial bijections of a set. This analogy shall be one of the driving forces of the upcoming chapters. Moreover, these locally injecitve dynamics present in the semigroup are suitable for the study of amenability and the relation with C*-algebras.

**Definition 3.3.1.** An *inverse semigroup* is a semigroup $S$ such that for every $s \in S$ there is a unique $s^* \in S$ satisfying $ss^*s = s$ and $s^*ss^* = s^*$.

Given an inverse semigroup $S$, the set $E(S) = \{s^*s \mid s \in S\}$ is the set of all *idempotents* (or *projections*) of $S$, i.e., elements satisfying $e = e^2 \in S$. Observe that in an inverse semigroup all idempotents are self-adjoint, i.e., they satisfy $e^* = e$. It is not obvious, though true nonetheless, that idempotents in an inverse semigroup commute with each other (see [110] or [58, Theorem 3]). Therefore $E(S)$ has a natural structure of a meet semi-lattice with respect to the partial order $e \leq f$ whenever $ef = fe = e$, and $S$ is a group if and only if $E(S)$ has exactly one element (the identity in the group). This partial order can, in addition, be extended to $S$ by stating that $s \leq t$ if and only if $ts^*s = s$.

We now introduce a couple of prototypical examples of inverse semigroups. For more (and more interesting) classes of examples see Sections 4.4 and 5.4.

**Example 3.3.2.** Given a discrete set $X$, we denote by $\mathcal{I}(X)$ the set of partial bijections of $X$, that is, its elements $(s, A, B) \in \mathcal{I}(X)$ are bijections $s\colon A \to B$, where $A, B \subset X$. The operation of the semigroup is just the composition of maps where it can be defined. Observe this semigroup contains both a zero element, namely $(0, \varnothing, \varnothing)$, and a unit, namely $(\mathrm{id}, X, X)$. Moreover, every inverse semigroup $S$ can be thought as contained in $\mathcal{I}(S)$ via the Wagner-Preston representation (see, e.g., [110, 81, 12] or Proposition 3.3.4 below).

**Example 3.3.3.** Given a positive integer $n \in \mathbb{N}$, the *policyclic monoid of rank* $n$ is the inverse monoid with presentation:

$$\mathcal{P}_n = \langle 1, 0, a_1, \ldots, a_n \mid a_i^* a_i = 1 \text{ and } a_i^* a_j = 0 \text{ for every } i, j = 1, \ldots, n \rangle.$$

These already appeared in works of Nivat and Perrot in [74], and then later in works of Cuntz [27]. After these, they have been extensively studied both within the semigroup literature (see, for instance, [33]) and the groupoid and C*-communities (see [88, 81, 36]). Of particular importance is the case when $n = 1$, where it is customary to consider it without the zero element:

$$\mathcal{T} = \langle a, a^* \mid a^*a = 1 \rangle,$$

which is known as the *bicyclic monoid*. From the point of view of functional analysis, however, this monoid would be called the *shift monoid*, while from the point of view of C*-algebras the natural name would be the *Toeplitz monoid*. See, moreover, Figure 2.3 for the Cayley graph of $\mathcal{T}$.

We next introduce the so-called *Wagner-Preston representation* of an inverse semigroup (see [110]), which will be of utmost importance in this thesis. To ease notation, given a discrete set $X$, we shall denote the set of partially defined bijections of $X$ by $\mathrm{PBij}(X)$. This notation is not commonly used in the literature, and we will use it only momentarily.

**Proposition 3.3.4.** *Let $S$ be a discrete inverse semigroup. Given $s \in S$ let*

$$D_{s^*s} := s^*sS = \{x \in S \mid s^*sx = x\} .$$

*Then the map*

$$v\colon S \to PBij(S) , \quad s \mapsto (v_s, D_{s^*s}, D_{ss^*}) , \quad \textit{where } v_s(x) = sx \ \textit{ when } x \in D_{s^*s}$$

*is an injective *-homomorphism of $S$.*

*Proof.* First note that $v_s$ indeed defines a partial bijection in $S$, since it is only defined whenever $x \in D_{s^*s}$. In addition, note that $v_s$ maps $D_{s^*s}$ bijectively onto $D_{ss^*}$. Indeed, for any pair $x, y \in D_{s^*s}$ such that $sx = sy$ we have that $x = s^*sx = s^*sy = y$ and, thus, $v_s \in \mathrm{PBij}(S)$. Moreover, note that

$$D_{(st)^*(st)} = \{x \in S \mid t^*s^*stx = x\} = t^* \cdot \{y \in S \mid s^*sy = y = tt^*y\} = t^* \cdot (D_{tt^*} \cap D_{s^*s}) .$$

Particularly, $v_{st}(x) = stx = v_s(tx) = v_s(v_t x)$ for any $x \in D_{(st)^*(st)}$, which implies $v_{st} = v_s v_t$ as partial bijections in $S$, i.e., $v$ defines a *-representation of $S$. Finally, $v$ is an injective map, since any two distinct elements $s \neq t$ must satisfy that either $s^*s \neq t^*t$ or $s^*s = t^*t$ and $s \neq t$. In the first case, and without loss of generality, let $x \in D_{s^*s} \setminus D_{t^*t}$. It then follows that $v_s(x) = sx$, while $v_t(x)$ is not defined. In the latter case, observe that $v_s(s^*s) = s \neq t = v_t(s^*s)$, proving that $v_s \neq v_t$. ☐

Figure 3.1 below depicts the Wagner-Preston representation of an inverse semigroup, and how the composition of maps $v_s, v_t$ works.

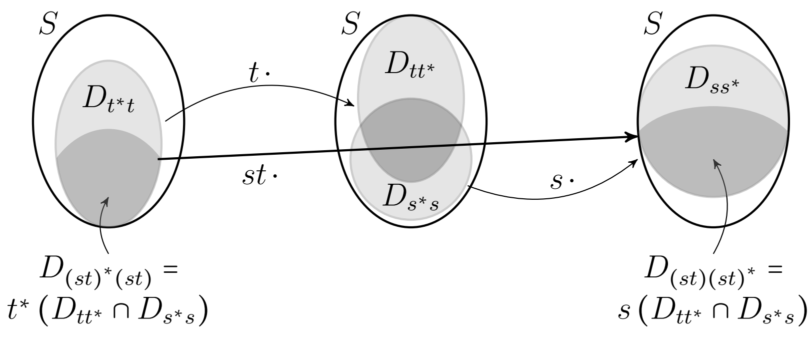


FIGURE 3.1: Rough picture of the Wagner-Preston representation of an inverse semigroup $S$. The partial translation $x \mapsto stx$ is defined as the composition of $s$ and $t$ whenever the composition makes sense.

As an immediate consequence of Proposition 3.3.4 we get the following, which introduces the *left regular representation* of a discrete inverse semigroup $S$.

**Proposition 3.3.5.** *Let $S$ be a discrete inverse semigroup. Then the map*

$$V\colon S \to \ell^2 S, \quad \text{where } V_s(\delta_x) := \begin{cases} \delta_{sx} & \text{if } x \in D_{s^*s} \\ 0 & \text{otherwise,} \end{cases}$$

*defines a *-representation of $S$ via partial isometries in $\ell^2 S$. Moreover, these partial isometries have commuting domains and ranges.*

Worth mentioning in the context of inverse semigroups are the special forms Green's relations take (see Section 2.4). Observe that $s \mathcal{L} t$ if $s^*s = t^*t$, that is, $s$ and $t$ are $\mathcal{L}$-related whenever $v_s$ and $v_t$ have the same domain as partially defined maps in $S$. Meanwhile, $s \mathcal{R} t$ whenever $ss^* = tt^*$, i.e., $v_s$ and $v_t$ have the same range in $S$ (namely $D_{ss^*} = D_{tt^*}$). The $\mathcal{H}$-relation then becomes $s \mathcal{H} t$ if $s^*s = t^*t$ and $ss^* = tt^*$. Finally, elements $s, t \in S$ are $\mathcal{D}$-related if there is some $r \in S$ such that $s^*s = r^*r$ and $rr^* = tt^*$.

We end the chapter mentioning that we believe the context of inverse semigroups is the most natural one when one wishes to study amenability-related notions in semigroups and their C*-algebras. On the one hand, and as we shall see, it allows for unexpected behavior, such as locally finite inverse semigroups (which are clearly amenable) that do not have Yu's property A (see Section 5.4.5). On the other, they are still tractable from a representation point of view, which will be of use when relating these notions with some C*-notions in the upcoming chapter.

# Chapter 4
# Inverse semigroups and their representations

After the discussion in Chapter 3 it seems plausible that, within the context of inverse semigroups, one might be able to reproduce the so-called *classical equivalences*, that is, produce a Følner condition and a paradoxicality condition that are equivalent to the amenability of the semigroup. We develop such a theory in this chapter.

The chapter is organized as follows. Section 4.1 introduces an alternative characterization for the amenability of an inverse semigroup. To this end, we introduce two independent conditions, named *domain-measurability* and the *localization of the measure*, whose combination is equivalent to the amenability of the semigroup. Section 4.2 then generalizes these notions to so-called representations (or actions) of inverse semigroups, and studies their relation in an algebraic scenario, meaning *non-C**. Later on, in Section 4.3, we recall the construction of the crossed product of a representation of an inverse semigroup and give the first approximation results of such C*-algebra in terms of (amenable) traces and proper infiniteness, and in relation to domain-measurability of the representation. Section 4.4 is devoted to give examples of some of these representations, while Section 4.5 relates the research here conducted with previous work on groupoids.

## 4.1 A characterization of invariant measures

Recall (see Day's original work [29] or Section 2.4.2) that a semigroup $S$ is *amenable* if there is a finitely additive probability measure $\mu\colon \mathcal{P}(S) \to [0,1]$ such that

$$\mu(A) = \mu\left(s^{-1}A\right), \quad \text{for all } s \in S \text{ and } A \subset S.$$

This definition, although perfectly adequate, has one main disadvantage, namely the dynamics involved. For instance, a non-amenable inverse semigroup does not immediately have elements $a_i, b_j \in S$ whose (left) regular representation induce a properly infinite projection. Note this shall be a key step in the proof of Theorem 4.3.7, one of the main theorems of the chapter. It is, therefore, our first goal to give an alternative characterization of the invariance of a measure in a way that only uses left multiplications and not preimages.

Before developing the new approach we recall that, within the semigroup literature, amenability in inverse semigroups has been amply studied. In particular, it is known that the amenability of an inverse semigroup is closely related to the amenability of its group homomorphic image $G(S)$, as the following result of Duncan and Namioka shows (see [31]).

**Theorem 4.1.1.** *A countable discrete inverse semigroup $S$ is amenable if and only if the group $G(S)$ is amenable, where $G(S) = S/\sigma$ and $s\sigma t$ if and only if $es = et$ for some projection $e \in E(S)$.*

In the literature the latter fact has led to the established opinion that amenability in the inverse semigroup case can be traced back to the group case. Our results later will refine this line of thought.

**Lemma 4.1.2.** *Let $S$ be an inverse semigroup. For any $s \in S$ and $A, B \subset S$ the following relations hold:*

*(i)* $s(s^{-1}A \cap s^*ss^{-1}A) = A \cap ss^*A = ss^{-1}A \subset A \subset s^{-1}sA$.

*(ii)* $ss^*(A \cap ss^*B) = A \cap ss^*B$.

*(iii)* $s^{-1}(A \setminus ss^*A) = \varnothing$.

*Proof.* The inclusions $ss^{-1}A \subset A \subset s^{-1}sA$ follow directly from the definition and (ii) is straightforward to check. To show $s(s^{-1}A \cap s^*ss^{-1}A) = A \cap ss^*A$ choose $t \in s^{-1}A \cap s^*ss^{-1}A$. Then $st \in A$ and $t = s^*sq$ for some $q \in S$ with $sq \in A$, hence $st = ss^*sq \in ss^*A$. To show the reverse inclusion consider $t \in A \cap ss^*A$, i.e., $A \ni t = ss^*a$ for some $a \in A$. Then $s^*a \in s^{-1}A$ and $t = ss^*s(s^*a) \in s(s^{-1}A \cap s^*ss^{-1}A)$. The remaining equalities are proved in a similar vein. □

**Theorem 4.1.3.** *Let $S$ be an inverse semigroup, and $\mu$ be a probability measure on it. The following conditions are equivalent:*

*(1)* $\mu$ *is invariant in the sense of Day, i.e.,* $\mu(A) = \mu(s^{-1}A)$ *for all* $s \in S$, $A \subset S$.

*(2)* $\mu$ *satisfies the following conditions for all* $s \in S$, $A \subset S$:

*(2.a)* $\mu(A) = \mu(A \cap s^*sA)$.

*(2.b)* $\mu(s^*sA) = \mu(sA)$.

*Proof.* For notational simplicity we show conditions (2.a) and (2.b) interchanging the roles of $s$ and $s^*$. The implication (1) ⇒ (2) follows from two simple observations. First, note that

$$\mu\left(A \setminus ss^*A\right) = \mu\left(s^{-1}\left(A \setminus ss^*A\right)\right) = \mu\left(\varnothing\right) = 0.$$

Therefore $\mu(A) = \mu(A \cap ss^*A) + \mu(A \setminus ss^*A) = \mu(A \cap ss^*A)$, as required. Secondly, observe that $s^*A \subset s^{-1}ss^*A$, and hence

$$\begin{aligned}\mu\left(ss^*A\right) = \mu\left(s^{-1}ss^*A\right) &= \mu\left(s^{-1}ss^*A \setminus s^*A\right) + \mu\left(s^*A\right)\\ &= \mu\left(\left(s^*\right)^{-1}\left(s^{-1}ss^*A \setminus s^*A\right)\right) + \mu\left(s^*A\right) = \mu\left(s^*A\right).\end{aligned}$$

The implication (2) ⇒ (1) follows from (2.a), (2.b) and Lemma 4.1.2 (i):

$$\mu\left(s^{-1}A\right) = \mu\left(s^{-1}A \cap s^*ss^{-1}A\right) = \mu\left(s\left(s^{-1}A \cap s^*ss^{-1}A\right)\right) = \mu\left(A \cap ss^*A\right) = \mu\left(A\right).$$

□

Observe that condition (2.a) in Theorem 4.1.3 is always satisfied in the group case. In the following easy corollary we combine conditions (2.a) and (2.b) in Theorem 4.1.3 into a single one.

**Corollary 4.1.4.** *Let $S$ be a countable and discrete inverse semigroup and $\mu$ be a probability measure on it. Then $\mu$ is invariant if and only if $\mu(A) = \mu(s(A \cap s^*sA))$ for all $s \in S$ and $A \subset S$.*

*Proof.* Assume that $\mu$ is invariant, hence satisfies conditions (2.a) and (2.b) in Theorem 4.1.3. Since $A \cap s^*sA \subset s^*sA$ we have

$$\mu\left(A\right) = \mu\left(A \cap s^*sA\right) = \mu\left(s^*s\left(A \cap s^*sA\right)\right) = \mu\left(s\left(A \cap s^*sA\right)\right).$$

To show the reverse implication we prove first condition (2.a), which follows from

$$\begin{aligned}\mu\left(A\right) = \mu\left(s\left(A \cap s^*sA\right)\right) &= \mu\Big(s^*\Big(s\left(A \cap s^*sA\right) \cap ss^*s\left(A \cap s^*sA\right)\Big)\Big) \\ &= \mu\left(s^*s\left(A \cap s^*sA\right)\right) = \mu\left(A \cap s^*sA\right),\end{aligned}$$

where for the last equation we used Lemma 4.1.2 (ii). Finally, condition (2.b) follows directly from $\mu(A) = \mu(s(A \cap s^*sA))$ just by replacing the set $A$ by $s^*sA$. □

The characterization of an invariant measure $\mu$ given in Theorem 4.1.3 means that $\mu$ is measurable (see Definition 2.4.2) via $s$ but only when the action of $s$ is restricted to its domain (namely $\mu(s^*sA) = \mu(sA)$). In addition, the measure of any set $A$ is localized *within the domain of every* $s \in S$ (namely $\mu(A) = \mu(A \cap s^*sA)$).

Observe that one of the main theorems of the chapter (see Theorem 4.3.7) states that the C*-algebra $\ell^\infty(S) \rtimes_r S$ (yet undefined) has an amenable trace if and only if $S$ has a finitely additive measure satisfying condition (2.b) in Theorem 4.1.3. This, and the fact that (2.a) and (2.b) are independent, justify the next definitions. Recall that $D_{s^*s} := \{x \in S \mid s^*sx = x\} = s^*sS$.

**Definition 4.1.5.** Let $S$ be a countable and discrete inverse semigroup and let $\mu{:}\,\mathcal{P}(S) \to [0,1]$ be a finitely probability additive measure. Then:

(1) Given a set $A \subset S$, we say $\mu$ is a *domain-measure for* $A$ if $\mu(A) = 1$ and $\mu(B) = \mu(sB)$ for all $s \in S$ and $B \subset D_{s^*s}$. Likewise, we say that $A$ is *domain-measurable* when such $\mu$ exists. If $A = S$ then we say $S$ is *domain-measurable.*

(2) We say $\mu$ is *localized* if $\mu(D_{s^*s}) = 1$ for all $s \in S$ (or, equivalently, $\mu(B) = \mu(B \cap D_{s^*s})$ for all $s \in S$ and $B \subset S$).

Domain measurable semigroups can be seen as a possible generalization of group amenability. To further specify this idea compare Theorems 4.3.7 and 2.1.5. See also Figure 4.1 below for an illustration of the domain-measurability and the localization of a probability measure.

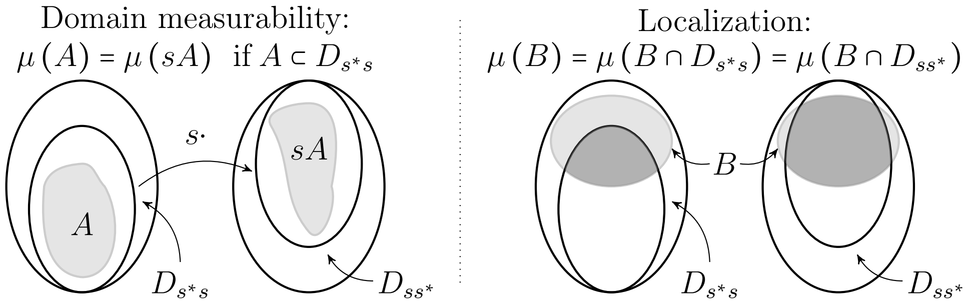


FIGURE 4.1: Graphic depiction of Day's invariance of a measure $\mu$.

**Example 4.1.6.** Every amenable semigroup is domain-measurable. Following the ideas behind Examples 2.1.8 and 2.2.10, we build a natural class of non-amenable, domain measurable semigroups. Let $A, N$ be disjoint inverse semigroups, with $A$ amenable and $N$ non-amenable. Consider then the semigroup $S = A \sqcup N$, where $an := n =: na$ for every $a \in A$, $n \in N$. It is routine to show $S$ is an inverse semigroup and we claim that it is non-amenable and domain measurable. Indeed, suppose it is amenable and let $\mu$ be an invariant measure on it. For any $n \in N$, $\mu(A) = \mu(n^{-1}A) = \mu(\varnothing) = 0$ and hence $\mu(N) = 1$. Therefore $\mu$ would restrict to an invariant mean on $N$, contradicting the hypothesis.

To prove now that $S$ is domain measurable, just choose an invariant measure $\nu$ on $A$ (that exists since $A$ is amenable) and extend it to $S$ as $\widehat{\nu}(A' \sqcup N') = \nu(A')$, for any $A' \subset A$ and $N' \subset N$. This measure will satisfy $\widehat{\nu}(s^*sB) = \widehat{\nu}(sB)$ for every $s \in S$ and $B \subset S$.

The main example in this class is the free group $\mathbb{F}_2$ with an extra unit 1 adjoined, namely $S := \{1\} \sqcup \mathbb{F}_2$. By the argument above $S$ is domain-measurable (letting $\mu = \delta_1$), but not amenable, since any invariant mean on $S$ restricts to an invariant mean on $\mathbb{F}_2$, which is a non-amenable group. In a sense one can say that $\mathbb{F}_2$ is *below* 1, since $1\omega = \omega$ for any $\omega \in \mathbb{F}_2$. This idea of *above* and *below*, or of behavior that *falls down*, will be a recurrent theme in the thesis (see, for instance, Section 5.1.2).

The following remarks place the domain-measure and localization conditions above in context within the semigroup literature.

**Remark 4.1.7.** Recall from [30] that a semigroup is called *fairly amenable* if it has a probability measure $\mu$ such that

$$\mu(A) = \mu(sA) \quad \text{if } s \text{ acts injectively on } A \subset S.$$

Observe that if $\mu$ satisfies the preceding condition then it is also a domain-measure for $S$, since $s$ acts injectively on $s^*sA \subset s^*sS = D_{s^*s}$. However, a domain measure satisfying the condition of domain measurability need not implement fair amenability. In fact, consider an inverse semigroup $S$ with a 0 element and some other non-zero element $s \in S$. Then $S$ is domain measurable, since it is amenable, with an invariant measure $\mu$ satisfying $\mu(\{0\}) = 1$. This measure, however, cannot implement fair amenability.

**Remark 4.1.8.** Argabright and Wilde (see [10] and Definition 2.4.2) introduced the so-called *strong Følner condition* based on previous work of Følner [39] and Namoika [71]. In this way, a semigroup satisfies the *strong Følner condition* if for every finite $\mathcal{F} \subset S$ and $\varepsilon > 0$ there is some finite $(\mathcal{F}, \varepsilon)$-Følner set $F \subset S$ such that

$$|sF| = |F|, \quad \text{for all } s \in \mathcal{F}.$$

This says that the set $F$ can be *localized* within a part of $S$ where $s \in \mathcal{F}$ acts injectively and, hence, the strong Følner property is reminiscent of the localization condition (2.a) in Theorem 4.1.3. To further justify this analogy, observe that Gray and Kambites proved in [42] that, within the class of semigroups satisfying the *Klawe condition* (see [55] and [42, Theorem 2.6]), the strong Følner condition actually characterizes the amenability of the semigroup.

## 4.2 Representations of inverse semigroups

This section studies the amenability notions introduced previously, and gives a relation between them. Moreover, the techniques here presented allow us to work within a slightly more general scenario, namely that of inverse semigroups *acting* on discrete sets. For now, let us recall what we mean by *representation* (or *action*) (see [35]).

**Definition 4.2.1.** Let $S$ be an inverse semigroup, and let $X$ be a discrete set. A *representation* (or an *action*) of $S$ on $X$ is a semigroup homomorphism $\alpha: S \to \mathcal{I}(X)$.

Observe that the homomorphism $v$ described in Proposition 3.3.4 is a representation of $S$ on $\mathcal{I}(S)$ in the sense above. Moreover, recall that $v$ is normally called the *Wagner-Preston* representation of $S$.

### 4.2.1 Their type semigroup

In order to give a relation between domain-measurability and the Følner condition (see Theorem 4.2.14) we will make use of the *type semigroup construction.* As was mentioned in the introduction, Tarski proved in [106] that a non-amenable group always has a paradoxical decomposition by means of constructing a semigroup, which has come to be known as the *type semigroup of* $G$, that captures the essence of a paradoxical decomposition of the group $G$. This construction has had great impact since, and has been used repeatedly in many different areas (see, e.g., [82, 89, 86, 108]). We now adapt this notion to the context of inverse semigroups.

**Definition 4.2.2.** Let $\alpha: S \to \mathcal{I}(X)$ be a representation of the inverse semigroup $S$ on a set $X$. We define the *type semigroup* $\mathrm{Typ}(\alpha)$ as the commutative monoid generated by symbols $[A]$ with $A \in \mathcal{P}(X)$ and relations

(1) $[\varnothing] = 0$.
(2) $[A] = [\alpha_s(A)]$ if $A \subset D_{s^*s}$.
(3) $[A \sqcup B] = [A] + [B]$ if $A \cap B = \varnothing$.

This definition allows to easily check if a map from $\mathrm{Typ}(\alpha)$ to another semigroup is a homomorphism. However, it may not be immediately obvious why this semigroup $\mathrm{Typ}(\alpha)$ is called a type semigroup. Indeed, Tarski originally used an apparently different approach.

**Definition 4.2.3.** Let $\alpha: S \to \mathcal{I}(X)$ be a representation of the inverse semigroup $S$ on a set $X$. We say $A, B \subset X$ are *equidecomposable*, and write $A \sim B$, if there are sets $A_i \subset A$ and elements $s_i \in S$, $i = 1, \ldots, n$, such that $A_i \subset D_{s_i^* s_i}$ for $i = 1, \ldots, n$, and

$$A = A_1 \sqcup \cdots \sqcup A_n \ \text{ and } \ \alpha_{s_1}(A_1) \sqcup \cdots \sqcup \alpha_{s_n}(A_n) = B.$$

The equi-decomposability relation given above realizes the idea that, up to finitely-many left multiplications by elements of $S$, the sets $A$ and $B$ are the same. Observe that, by definition, given any domain-measure $\mu$ on $S$ and two sets $A, B \subset S$, if $A$ and $B$ are equidecomposable then $\mu(A) = \mu(B)$.

It is routine to show that $\sim$ above is an equivalence relation. Indeed, note that since $1 \in S$ we have $A \sim A$. Furthermore, the relation $\sim$ is clearly symmetric by choosing $B_i := \alpha_{s_i}(A_i)$ and the dynamics $t_i := s_i^*$. Finally, if $A \sim B \sim C$ then there are $A_i, B_j \subset X$ and $s_i, t_j \in S$ such that $A_i \subseteq D_{s_i^* s_i}$, $B_j \subseteq D_{t_j^* t_j}$, and

$$A = A_1 \sqcup \cdots \sqcup A_n \ \text{ and } \ \alpha_{s_1}(A_1) \sqcup \cdots \sqcup \alpha_{s_n}(A_n) = B,$$
$$B = B_1 \sqcup \cdots \sqcup B_m \ \text{ and } \ \alpha_{t_1}(B_1) \sqcup \cdots \sqcup \alpha_{t_m}(B_m) = C.$$

In this case the sets $A_{ij} = \alpha_{s_i^*}(\alpha_{s_i}(A_i) \cap B_j)$ and the elements $r_{ij} = t_j s_i$ implement the relation $A \sim C$. Now, given a representation $\alpha: S \to \mathcal{I}(X)$, consider the following extensions:

- The semigroup $S \times \mathrm{Perm}(\mathbb{N})$, where $\mathrm{Perm}(\mathbb{N})$ is the finite permutation group of $\mathbb{N}$, that is the group of permutations moving only a finite number of elements.
- A set $A \subset X \times \mathbb{N}$ is called *bounded* if $A \subset X \times F$, with $F \subset \mathbb{N}$ finite.

Then there is an obvious representation of $S \times \mathrm{Perm}$ on $X \times \mathbb{N}$ given coordinate-wise, which will be also denoted by $\alpha$. Hence, it makes sense to ask when two bounded sets $A, B \subset X \times \mathbb{N}$ are equidecomposable. Define

$$\widehat{X} := \{A \subset X \times \mathbb{N} \mid A \text{ is bounded}\} / \sim .$$

This set has the natural structure of a commutative monoid with zero element $0 = \varnothing$ and sum defined as follows. Given two bounded sets $A, B \subset X \times \mathbb{N}$, let $k \in \mathbb{N}$ be such that $A \cap B' = \varnothing$, where $B' := \{(b, n+k) \mid (b,n) \in B\}$. Then define $[A] + [B] := [A \sqcup B']$. One can verify that + is well-defined, associative and commutative. We only need the notation $\widehat{X}$ temporarily and after the proof of the following result we will only use the symbol $\mathrm{Typ}(\alpha)$.

**Proposition 4.2.4.** *Let $\alpha\colon S \to \mathcal{I}(X)$ be a representation of a unital inverse semigroup $S$. Then the map*

$$\gamma\colon \mathit{Typ}(\alpha) \longrightarrow \widehat{X}, \quad \textit{where } \gamma([A]) = [A \times \{1\}]$$

*is a monoid isomorphism.*

*Proof.* Since $[A \times \{1\}] = [A \times \{i\}]$ in $\widehat{X}$, we easily see that this map is well-defined and surjective. To show it is injective assume that $\gamma(\sum_{i=1}^{n}[A_i]) = \gamma(\sum_{j=1}^{m}[B_j])$ for subsets $A_i, B_j$ of $X$. Then

$$A := \bigsqcup_{i=1}^{n} A_i \times \{i\} \sim \bigsqcup_{j=1}^{m} B_j \times \{j\} =: B,$$

and so, by definition, there are subsets $W_1, \ldots, W_l$ of $X$, and numbers $n_1, \ldots, n_l$, $m_1, \ldots, m_l \in \mathbb{N}$ and elements $s_1, \ldots, s_l \in S$ such that $W_k \subset D_{s_k^* s_k}$ for $k = 1, \ldots, l$ and

$$A = \bigsqcup_{k=1}^{l} W_k \times \{n_k\}, \quad B = \bigsqcup_{k=1}^{l} \alpha_{s_k}(W_k) \times \{m_k\}.$$

It follows that there is a partition $\{1, \ldots, l\} = \bigsqcup_{i=1}^{n} I_i$ such that for each $i \in \{1 \ldots, n\}$ we have $A_i = \bigsqcup_{j \in I_i} W_j$. We thus get in $\mathrm{Typ}(\alpha)$

$$\sum_{i=1}^{n}[A_i] = \sum_{i=1}^{n}\sum_{j\in I_i}[W_j] = \sum_{i=1}^{n}\sum_{j\in I_i}[\alpha_{s_j}(W_j)] = \sum_{j=1}^{l}[\alpha_{s_j}(W_j)] = \sum_{j=1}^{m}[B_j],$$

showing injectivity. □

For simplicity we will often denote $\alpha_s(x) \in X$ by $sx$ and $sA$ will stand for $\alpha_s(A)$ for any $s \in S$, $x \in X$ and $A \subset X$. Recall that $sx$ is defined only if $x \in D_{s^*s}$. We extend next Definition 4.1.5 above to representations.

**Definition 4.2.5.** Let $\alpha\colon S \to \mathcal{I}(X)$ be a representation of $S$ and let $A \subset X$.

(1) The set $A$ is *$S$-domain measurable* if there is a measure $\mu\colon \mathcal{P}(X) \to [0, \infty]$ normalized at $A$ (i.e., $\mu(A) = 1$) such that $\mu(B) = \mu(sB)$ for all $s \in S$ and

$B \subset D_{s^*s}$. We say that $X$ is *$S$-domain measurable* when the latter holds for $A = X$.

(2) The set $A$ is *$S$-domain Følner* if there is a sequence $\{F_n\}_{n \in \mathbb{N}}$ of finite, non-empty subsets of $A$ such that for all $s \in S$

$$\frac{|s(F_n \cap D_{s^*s}) \setminus F_n|}{|F_n|} \xrightarrow{n \to \infty} 0$$

(3) The set $A$ is *$S$-paradoxical* if there are $A_i, B_j \subset X$ and $s_i, t_j \in S$, $i = 1, \dots, n$, $j = 1, \dots, m$, such that $A_i \subseteq D_{s_i^* s_i}$, $B_j \subseteq D_{t_j^* t_j}$ and

$$\begin{aligned} A &= s_1 A_1 \sqcup \dots \sqcup s_n A_n = t_1 B_1 \sqcup \dots \sqcup t_m B_m \\ &\supset A_1 \sqcup \dots \sqcup A_n \sqcup B_1 \sqcup \dots \sqcup B_m. \end{aligned}$$

Recall that, in a commutative semigroup $\mathcal{S}$, we denote by $n \cdot \beta$ the sum $\beta + \dots + \beta$ of $n$ terms. Also, the only (pre-)order that we use on $\mathcal{S}$ is the so-called *algebraic pre-order*, defined by $x \le y$ if and only if $x + z = y$ for some $z \in \mathcal{S}$.

**Remark 4.2.6.** Note that $S$ is *domain measurable* in the sense of Definition 4.1.5 precisely when $S$ is $S$-domain measurable with respect to the canonical Wagner-Preston representation $\alpha: S \to \mathcal{I}(S)$. Note also that $A \subset X$ being paradoxical is the same as saying that $2[A] \le [A]$ in $\mathrm{Typ}(\alpha)$.

**Lemma 4.2.7.** *Let $\alpha: S \to \mathcal{I}(X)$ be a representation of $S$, and consider the type semigroup $Typ(\alpha)$ constructed above. Then the following hold:*

*(1) For any bounded sets $A, B \subset X \times \mathbb{N}$ if $A \sim B$ then there is a bijection $\phi: A \to B$ such that for any $C \subset A$ and $D \subset B$ one has $C \sim \phi(C)$ and $D \sim \phi^{-1}(D)$.*

*(2) For any $[A], [B] \in Typ(\alpha)$, if $[A] \le [B]$ and $[B] \le [A]$, then $[A] = [B]$.*

*(3) A subset $A$ of $X$ is $S$-paradoxical if and only if $[A] = 2 \cdot [A]$.*

*(4) For any $[A], [B] \in Typ(\alpha)$ and $n \in \mathbb{N}$, if $n \cdot [A] = n \cdot [B]$, then $[A] = [B]$.*

*(5) If $[A] \in Typ(\alpha)$ and $(n+1) \cdot [A] \le n \cdot [A]$ for some $n \in \mathbb{N}$, then $[A] = 2 \cdot [A]$.*

*Proof.* The proof of this lemma is virtually the same as in the group case (see, e.g., [92, p. 10]). For convenience of the reader we include a sketch of the proofs.

(1) Define $\phi: A \to B$ by multiplication by $s_i$ in each of the subsets $A_i$, where $A = \sqcup_i A_i$ and $B = \sqcup_i s_i A_i$.

(2) By hypothesis there are $[A_0], [B_0] \in \mathrm{Typ}(\alpha)$ such that $[A] + [A_0] = [B]$ and $[B] + [B_0] = [A]$. In addition, observe that, without loss of generality, we can suppose that $A \cap A_0 = \varnothing = B \cap B_0$. Choose $\phi: A \sqcup A_0 \to B$ and $\psi: B \sqcup B_0 \to A$ as in (1) and consider

$$C_0 := A_0,\ C_{n+1} := \psi(\phi(C_n)) \text{ and } C := \cup_{n=0}^{\infty} C_n.$$

It then follows that $(B \sqcup B_0) \smallsetminus \phi(C) = \psi^{-1}(A \smallsetminus C) = \psi^{-1}(A \sqcup A_0 \smallsetminus C)$ and hence

$$A \sqcup A_0 = (A \smallsetminus C) \sqcup C \sim \psi^{-1}(A \smallsetminus C) \sqcup \phi(C) = (B \sqcup B_0 \smallsetminus \phi(C)) \sqcup \phi(C) = B \sqcup B_0.$$

Therefore $B \sim A \sqcup A_0 \sim B \sqcup B_0 \sim A$.

Now (3) follows from the definitions and (2).

Claim (4) uses graph theory and follows from König's Theorem (see [82, Theorem 0.2.4]). If $n \cdot [A] = n \cdot [B]$ then there are sets $A_i, B_j$ with the following properties:

(a) $A_1, \ldots, A_n$ are pairwise disjoint, just as $B_1, \ldots, B_n$.
(b) $n \cdot [A] = [A_1] + \cdots + [A_n] = [B_1] + \cdots + [B_n] = n \cdot [B]$.
(c) For every $i = 1, \ldots, n$ we have $A_i \sim A$ and $B_i \sim B$.

Consider then the bijections $\phi_j \colon A_1 \to A_j$, $\psi_j \colon B_1 \to B_j$ and $\chi \colon n \cdot [A] \to n \cdot [B]$ induced by $\sim$ as in (1). For $a \in A_1$ denote by $\overline{a}$ the set $\{\phi_1(a), \ldots, \phi_n(a)\}$ (and analogously for $b \in B_1$). Consider now the bipartite graph defined by:

- Its sets of vertices are $X = \{\overline{a} \mid a \in A_1\}$ and $Y = \left\{\overline{b} \mid b \in B_1\right\}$.
- The vertices $\overline{a}$ and $\overline{b}$ are joined by an edge if $\chi(\phi_j(a)) \in \overline{b}$ for some $j = 1, \ldots, n$.

Then this graph is $n$-regular and, by König's Theorem, it has a perfect matching $F$. In this case it can be checked that the sets

$$\begin{aligned} C_{j,k} &:= \left\{a \in A_1 \mid \exists b \in B_1 \text{ such that } \left(\overline{a}, \overline{b}\right) \in F \text{ and } \chi(\phi_j(a)) = \psi_k(b)\right\}, \\ D_{j,k} &:= \left\{b \in B_1 \mid \exists a \in A_1 \text{ such that } \left(\overline{a}, \overline{b}\right) \in F \text{ and } \chi(\phi_j(a)) = \psi_k(b)\right\}, \end{aligned}$$

are respectively a partition of $A_1$ and $B_1$. Furthermore $\psi_k^{-1} \circ \chi \circ \phi_j$ is a bijection from $C_{j,k}$ to $D_{j,k}$ implementing the relations $C_{j,k} \sim D_{j,k}$. These, in turn, implement $A \sim A_1 \sim B_1 \sim B$.

(5) follows from (2) and (4). Indeed, from the hypothesis

$$2 \cdot [A] + n \cdot [A] = (n+1) \cdot [A] + [A] \leq n \cdot [A] + [A] = (n+1) \cdot [A] \leq n \cdot [A].$$

Iterating this argument we get $2n \cdot [A] \leq n \cdot [A]$ and, since the other inequality trivially holds, $n \cdot [A] = 2n \cdot [A]$. Applying (4) we conclude that $[A] = 2 \cdot [A]$. □

To end the section we recall one of Tarski's fundamental results [106] (see also [92, Theorem 0.2.10]). This will be used in the proof of Theorem 4.2.14.

**Theorem 4.2.8.** *Let $(\mathcal{S}, +)$ be a commutative semigroup with neutral element 0 and let $\epsilon \in \mathcal{S}$. The following are then equivalent:*

*(i)* $(n+1) \cdot \epsilon \nleq n \cdot \epsilon$ *for all* $n \in \mathbb{N}$.
*(ii) There is a semigroup homomorphism* $\nu \colon (\mathcal{S}, +) \to ([0, \infty], +)$ *with* $\nu(\epsilon) = 1$.

### 4.2.2 A Følner characterization and Namioka's trick

In order to prove the main result of the section (see Theorem 4.2.14) we need to introduce the following action of the inverse semigroup $S$, which will naturally lead to the behavior of domain measures as functionals on $\ell^\infty(X)$. These notions, in turn, will replicate the so-called *Namioka's trick* (see [71, 26]). Given a representation $\alpha\colon S \to \mathcal{I}(X)$ and $s \in S$ let

$$E_{s^*s} := \{f \in \ell^\infty(X) \text{ such that } \operatorname{supp}(f) \subset D_{s^*s}\},$$

and for any $f \in E_{s^*s}$ consider

$$(sf)(x) := f(s^*x)\, p_{ss^*}(x) = \begin{cases} f(s^*x) & \text{if } x \in D_{ss^*} \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where $p_{ss^*}$ denotes the characteristic function of $D_{ss^*}$. Recall that $s^*x$ is not defined unless $x \in D_{ss^*}$ and, hence, the term $p_{ss^*}$ appears in Eq. (4.1) to avoid a slight abuse of notation. However, we will normally drop it and simply consider $f(s^*\cdot)$ as constantly 0 outside of $D_{ss^*}$. In addition, note this allows us to extend the action of $s$ to all of $\ell^\infty(X)$ via $(sf)(x) = f(s^*x) \cdot p_{ss^*}(x)$. This will simplify our notation in some lemmas, such as Lemma 4.2.11.

**Proposition 4.2.9.** *Let $\alpha\colon S \to \mathcal{I}(X)$ be a representation of a discrete inverse semigroup. Then Eq. (4.1) above defines an action of $S$ on $\ell^\infty(X)$.*

*Proof.* Once the notation is clear the proof is elementary. Simply note that the sets $E_{s^*s} \subset \ell^\infty(X)$ are closed two-sided ideals. In addition, for instance, if $sf = sg$ for some $f, g \in E_{s^*s}$ then $f = ss^*f = ss^*g = g$. The rest of the properties are proved in a similar way. □

The next result uses the action in Eq. (4.1) and establishes an invariance condition in the context of states on $\ell^\infty(X)$.

**Proposition 4.2.10.** *Let $\alpha\colon S \to \mathcal{I}(X)$ be a representation of $S$. If $\mu$ is domain measure for $X$ (cf., Definition 4.1.5), then there is a state $m\colon \ell^\infty(X) \to \mathbb{C}$ such that for every $s \in S$*

$$m(sf) = m(f) \quad \text{for all } f \in E_{s^*s} \subset \ell^\infty(X).$$

*Proof.* For a set $B \subset S$ define $m(p_B) := \mu(B)$, where $p_B$ denotes the characteristic function on $B$, and extend by linearity to simple functions and by continuity to all $\ell^\infty(X)$. Then $m(sf) = m(f)$ for all $f \in E_{s^*s}$ if and only if $m(sp_B) = m(p_B)$ for any $p_B \in E_{s^*s}$, i.e., for any $B \subset D_{s^*s}$. Observe this follows from the fact that $sp_B = p_{sB}$ and the domain-measurability of $\mu$:

$$m(sp_B) = m(p_{sB}) = \mu(sB) = \mu(B) = m(p_B)$$

since, by assumption, $B \subset D_{s^*s}$. □

We next observe that the functional $m$ in the latter proposition can be approximated by functionals of finite support. Recall that if $h \in \ell^1(X) \subset \ell^\infty(X)$ then $s^*h(x) = h(sx)p_{s^*s}(x)$. In particular $s^*sh = hp_{s^*s}$.

**Lemma 4.2.11.** *Let $\alpha: S \to \mathcal{I}(X)$ be a representation of $S$. If $X$ is domain measurable then for every $\varepsilon > 0$ and finite $\mathcal{F} \subset S$ there is a positive $h \in \ell^1(X)$ of norm one and finite support such that $\|s^*h - s^*sh\|_1 < \varepsilon$ for every $s \in \mathcal{F}$.*

*Proof.* We denote by $\Omega$ the set of positive $h \in \ell^1(X)$ of norm one and finite support. By Proposition 4.2.10 there is a functional $m: \ell^\infty(X) \to \mathbb{C}$ such that $m(sf) = m(f)$ for every $f \in E_{s^*s}$ and $s \in S$. Note $f = s^*sf$, for $f \in E_{s^*s}$. Therefore, and since the normal states are weak-* dense in $(\ell^\infty(X))^*$, there is a net $\{h_\lambda\}_{\lambda\in\Lambda}$ in $\Omega$ such that

$$|\phi_{h_\lambda}(sf) - \phi_{h_\lambda}(f)| = |\phi_{s^*h_\lambda}(f) - \phi_{s^*sh_\lambda}(f)| \to 0, \tag{4.2}$$

where $\phi_h(f) = \sum_{x\in X} h(x)f(x)$ for $h \in \Omega, f \in \ell^\infty(X)$.

In order to transform the latter weak convergence to norm convergence we use a variation of a standard technique (see [26, 29, 71]). Consider the space $(\ell^1(X))^S$, which, when equipped with the product topology, is a locally convex linear topological space. Consider the map

$$T: \ell^1(X) \to \left(\ell^1(X)\right)^S, \quad h \mapsto (s^*h - s^*sh)_{s\in S}.$$

As the weak topology coincides with the product of weak topologies on $(\ell^1(X))^S$, it follows from Eq. (4.2) that 0 belongs to the weak closure of $T(\Omega)$. Furthermore, since $E$ is locally convex and $T(\Omega)$ is convex, its closure in the weak topology and in the product of the norm topologies are the same. Thus there is a (possibly different) net $\{h'_\lambda\}_{\lambda\in\Lambda}$ such that $\|s^*h'_\lambda - s^*sh'_\lambda\|_1 \to 0$ for all $s \in S$, which completes the proof. □

The following known result gives a useful description of functions in $\ell^1(X)_+$ of finite support. We give the proof of the following for the convenience of the reader.

**Lemma 4.2.12.** *Let $X$ be a set. Any $h \in \ell^1(X)_+$ of norm 1 and finite support can be expressed as*

$$h = (\beta_1/|A_1|)\, p_{A_1} + \dots + (\beta_N/|A_N|)\, p_{A_N}$$

*for some finite $A_1 \supset A_2 \supset \dots \supset A_N$, where $\beta_i \geq 0$ and $\sum_{i=1}^N \beta_i = 1$.*

*Proof.* Let $0 =: a_0 < a_1 < \dots < a_N$ be the distinct values of the function $h$. Then, defining $A_i := \{x \in X \mid a_i \leq h(x)\}$ we have that $A_1 \supset A_2 \supset \dots \supset A_N$. Furthermore $h = \sum_{i=1}^N \gamma_i p_{A_i}$, where $\gamma_i = a_i - a_{i-1}$ for $i \geq 1$. To conclude the proof put $\beta_i := \gamma_i|A_i|$, $i = 1, \dots, N$, and note that $\|h\|_1 = 1$ implies $\sum_{i=1}^N \beta_i = 1$. □

**Lemma 4.2.13.** *Let $\alpha: S \to \mathcal{I}(X)$ be a representation of $S$ and consider $s \in S$, $A \subset X$. Then $(s^*p_A - s^*sp_A)(x) < 0$ if and only if $x \in A \cap D_{s^*s} \smallsetminus s^*(A \cap D_{ss^*})$.*

*Proof.* By definition of the action given in Eq. (4.1) we compute

$$(s^*p_A - s^*sp_A)(x) = \begin{cases} 1 & \text{if} \quad x \in D_{s^*s} \smallsetminus A \text{ and } sx \in A \\ -1 & \text{if} \quad x \in A \cap D_{s^*s} \text{ and } sx \notin A \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if $(s^*p_A - s^*sp_A)(x) < 0$ then $x \in A \cap D_{s^*s} \smallsetminus s^*(A \cap D_{ss^*})$. The other implication is clear. □

We can now establish the main theorem of the section, characterizing the domain measurable representations of an inverse semigroup.

**Theorem 4.2.14.** *Let $S$ be a countable and discrete inverse semigroup with identity $1 \in S$ and $\alpha: S \to \mathcal{I}(X)$ be a representation of $S$ on $X$. The following are then equivalent:*

*(1) $X$ is $S$-domain measurable.*

*(2) $X$ is not $S$-paradoxical.*

*(3) $X$ is $S$-domain Følner.*

*Proof.* (1) ⇒ (2). Suppose $X$ is $S$-paradoxical. Then, choosing a domain measure $\mu$ and an $S$-paradoxical decomposition of $X$ we have

$$\begin{aligned} 1 = \mu(X) &\geq \mu(A_1) + \cdots + \mu(A_n) + \mu(B_1) + \cdots + \mu(B_m) \\ &= \mu\big(s_1 A_1 \sqcup \cdots \sqcup s_n A_n\big) + \mu\big(t_1 B_1 \sqcup \cdots \sqcup t_m B_m\big) = \mu(X) + \mu(X) = 2, \end{aligned}$$

which gives a contradiction.

(2) ⇒ (1). Consider the type semigroup $\mathrm{Typ}(\alpha)$ of the action. As $X$ is not $S$-paradoxical we know that $[X]$ and $2[X]$ are not equal in $\mathrm{Typ}(\alpha)$. It follows from Lemma 4.2.7 (5) that $(n+1)[X] \nleq n[X]$ and hence, by Tarski's Theorem 4.2.8, there exists a semigroup homomorphism $\nu: \mathrm{Typ}(\alpha) \to [0, \infty]$ such that $\nu([X]) = 1$. Then we define $\mu(B) := \nu([B])$, which satisfies $\mu(X) = 1$ and $\mu(B) = \mu(sB)$ for every $B \subset D_{s^*s}$, proving that $X$ is $S$-domain measurable.

(3) ⇒ (1). Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence witnessing the $S$-domain Følner property of $X$ and let $\omega \in \beta\mathbb{N} \smallsetminus \mathbb{N}$ be a free ultrafilter on $\mathbb{N}$. Consider the measure $\mu$ defined by

$$\mu(B) := \lim_{n \to \omega} |B \cap F_n| / |F_n|.$$

It follows from $\omega$ being an ultrafilter that $\mu$ is a finitely additive measure. Thus it remains to prove that $\mu(B) = \mu(sB)$ for any $B \subset D_{s^*s}$. Observe first that $s$ acts injectively on $B$. Therefore we have

$$\begin{aligned} |B \cap F_n| &= |s(B \cap F_n)| = |sB \cap s(F_n \cap D_{s^*s})| \\ &= |sB \cap s(F_n \cap D_{s^*s}) \cap F_n| + |(sB \cap s(F_n \cap D_{s^*s})) \smallsetminus F_n| \\ &\leq |sB \cap F_n| + |s(F_n \cap D_{s^*s}) \smallsetminus F_n|, \end{aligned}$$

and hence, normalizing by $|F_n|$ and taking ultralimits on both sides, we obtain $\mu(B) \leq \mu(sB)$. The other inequality follows from a similar argument, noting that

$$|s^* (sB \cap F_n)| = |sB \cap F_n|$$

since $s^*$ acts injectively on $sB$.

(1) ⇒ (3). To prove this implication we will refine Namioka's trick (see [71, Theorem 3.5]). By Lemma 4.2.11 and the fact that $X$ is domain measurable we conclude that for every $\varepsilon > 0$ and finite $\mathcal{F} \subset S$ (which we assume to be symmetric, i.e., $\mathcal{F} = \mathcal{F}^*$) there is a positive function $h \in \ell^1(X)$ of norm 1 and with finite support such that $\|s^*h - s^*sh\|_1 < \varepsilon/|\mathcal{F}|$ for all $s \in \mathcal{F}$. Moreover, by Lemma 4.2.12, we may express the function $h$ as a linear combination

$$h = \frac{\beta_1}{|A_1|} p_{A_1} + \cdots + \frac{\beta_N}{|A_N|} p_{A_N}, \text{ where } A_1 \supset A_2 \supset \cdots \supset A_N \text{ and } \sum_{i=1}^{N} \beta_i = 1.$$

Consider now the set $B_s := \cup_{i=1}^N (A_i \cap D_{s^*s}) \setminus s^*(A_i \cap D_{ss^*})$. By Lemma 4.2.13, the function $s^*h - s^*sh$ is non-negative on $X \setminus B_s$ and hence

$$\begin{aligned} \frac{\varepsilon}{|\mathcal{F}|} > \|s^*h - s^*sh\|_1 &\geq \sum_{x \in X \setminus B_s} s^*h(x) - s^*sh(x) \\ &= \sum_{i=1}^{N} \frac{\beta_i}{|A_i|} \left( \sum_{x \in X \setminus B_s} \left( s^* p_{A_i}(x) - s^* s p_{A_i}(x) \right) \right) \\ &\geq \sum_{i=1}^{N} \frac{\beta_i}{|A_i|} \left( \sum_{x \in s^*(A_i \cap D_{ss^*}) \setminus A_i \cap D_{s^*s}} \left( s^* p_{A_i}(x) - s^* s p_{A_i}(x) \right) \right). \end{aligned} \tag{4.3}$$

Observe that the last inequality follows from the fact that the sets $A_i$ are nested. Indeed, writing $Z_i = A_i \cap D_{s^*s}$ and $T_i = s^*(A_i \cap D_{ss^*})$ and denoting by $Y^c$ the complement in $X$ of a subset $Y$, we need to show that

$$Z_i^c \cap T_i \subset B_s^c = \cap_{j=1}^N (Z_j^c \cup T_j). \tag{4.4}$$

Now, if $i \geq j$ then $A_i \subset A_j$ and so $T_i \subset T_j$ which implies that $Z_i^c \cap T_i \subset Z_j^c \cup T_j$. If $i < j$ then $A_j \subset A_i$ and so $Z_j \subset Z_i$ which implies $Z_i^c \subset Z_j^c$ and so $Z_i^c \cap T_i \subset Z_j^c \cup T_j$. This shows (4.4) and, consequently, Eq. (4.3). Following the computations of Eq. (4.3):

$$\frac{\varepsilon}{|\mathcal{F}|} > \sum_{i=1}^{N} \beta_i \frac{|s^*(A_i \cap D_{ss^*}) \setminus A_i \cap D_{s^*s}|}{|A_i|} = \sum_{i=1}^{N} \beta_i \frac{|s^*(A_i \cap D_{ss^*}) \setminus A_i|}{|A_i|}. \tag{4.5}$$

The rest of the proof is similar to [71]. Denoting by $I = \{1, \ldots, N\}$, consider the measure on $I$ given by $\mu(J) = \sum_{j \in J} \beta_j$ for every $J \subset I$ and put $\mu(\varnothing) := 0$. For $s \in \mathcal{F}$ consider the set

$$K_s := \{ i \in I \text{ such that } |s(A_i \cap D_{s^*s}) \setminus A_i| < \varepsilon |A_i| \}.$$

From Eq. (4.5) it follows that

$$\varepsilon/|\mathcal{F}| > \sum_{i=1}^{N} \beta_i \, |s^*(A_i \cap D_{ss^*}) \setminus A_i| / |A_i| \geq \varepsilon \sum_{i \in I \setminus K_{s^*}} \beta_i = \varepsilon \, \mu \,(I \setminus K_{s^*})$$

and, thus, $\mu\,(I \setminus K_{s^*}) < 1/|\mathcal{F}|$. From this and $\mathcal{F} = \mathcal{F}^*$ we obtain

$$1 - \mu\,(\cap_{s\in\mathcal{F}} K_s) = \mu\,(I \setminus \cap_{s\in\mathcal{F}} K_s) = \mu\,(\cup_{s\in\mathcal{F}} I \setminus K_s) \leq \sum_{s\in\mathcal{F}} \mu\,(I \setminus K_s) < 1.$$

Therefore the set $\cap_{s\in\mathcal{F}} K_s$ is not empty since its measure is non-zero; any index $i_0 \in \cap_{s\in\mathcal{F}} K_s$ we will have that the corresponding set $A_{i_0}$ satisfies the domain Følner condition. □

**Remark 4.2.15.** We mention that the theory of type semigroups for representations of inverse semigroups includes the corresponding theory for partial actions of groups. Given a (discrete) group $G$ and a non-empty set $X$, Exel defines the notion of a *partial action of $G$ on $X$* (see Section 4.4.3 below). In this context one can associate in a natural way the type semigroup $\mathrm{Typ}(X, G)$ to the given partial action (see, e.g., [5, Section 7]). Moreover, in [35] Exel associates to each group $G$ an inverse semigroup $\mathcal{S}(G)$ such that the partial actions of $G$ on $X$ are in bijective correspondence with the representations $\alpha: \mathcal{S}(G) \to \mathcal{I}(X)$. Note that representations of inverse semigroups are called *actions* in [35]. In this context, it can be shown, using the abstract definitions of these semigroups, that the type semigroup $\mathrm{Typ}(\alpha)$ introduced in Definition 4.2.2 is naturally isomorphic to the type semigroup $\mathrm{Typ}(X, G)$ of the corresponding partial action of $G$ on $X$.

### 4.2.3 Localization and amenable actions

Day's definition of amenability, by Theorem 4.1.3, is equivalent to the semigroup having a localized domain-measure. The goal of this section, hence, is to prove the analogue of Theorem 4.2.14 but considering amenable representations instead of the weaker notion of domain measurable ones. We will have to refine the reasonings of the previous section including the localization condition

$$\mu\,(A \cap s^* s A) = \mu\,(A)\ , \quad s \in S\,,\ A \subset S.$$

We first extend this definition to the context of representations.

**Definition 4.2.16.** Let $\alpha: S \to \mathcal{I}(X)$ be a representation of the inverse semigroup $S$ and let $A \subset X$ be a subset. Then $A$ is *$S$-amenable* when there is a measure $\mu: \mathcal{P}(X) \to [0, \infty]$ such that:

(1) $\mu(A) = 1$.
(2) $\mu(B) = \mu(\alpha_s(B))$ for all $s \in S$ and $B \subset D_{s^*s}$.
(3) $\mu(B) = \mu(B \cap D_{t^*t})$ for all $t \in S$ and $B \subset X$.

We say that $X$ is $S$-*amenable* when the latter holds for $A = X$.

We begin with the following useful lemma.

**Lemma 4.2.17.** *Every countable inverse semigroup $S$ has a decreasing sequence of projections $\{e_n\}_{n\in\mathbb{N}}$ that is eventually below every other projection, that is, $e_n \geq e_{n+1}$ and for every $f \in E(S)$ there is some $n_0 \in \mathbb{N}$ such that $f \geq e_{n_0}$.*

*Proof.* Since $S$ is countable we can enumerate the set of projections $E(S) = \{f_1, f_2, \dots\}$. The lemma follows by letting $e_n := f_1 \dots f_n$. □

The localization property of the measure can be included in the reasoning leading to Theorem 4.2.14, yielding the following.

**Theorem 4.2.18.** *Let $S$ be a countable and discrete inverse semigroup with identity $1 \in S$ and $\alpha: S \to \mathcal{I}(X)$ be a representation of $S$ on $X$. Then the following conditions are equivalent:*

*(1) $X$ is $S$-amenable.*
*(2) $D_e$ is not $S$-paradoxical for any $e \in E(S)$.*
*(3) For every $\varepsilon > 0$ and finite $\mathcal{F} \subset S$ there is a finite non-empty $F \subset X$ such that $F \subset D_{s^*s}$ and $|sF \setminus F| < \varepsilon |F|$ for all $s \in \mathcal{F}$.*

*Proof.* Observe the equivalence between (1) and (2) follows from Theorem 4.2.14 and Lemma 4.2.17. Indeed, one can check that the proof of (1) ⇔ (2) in Theorem 4.2.14 works for any subset $A \subset X$, in particular if $A = D_e$. Thus, if $D_e$ is not paradoxical for any projection $e$ then there are measures $\mu_e$ on $X$ such that $\mu_e(D_e) = 1$. Now, by Lemma 4.2.17, let $\{e_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of projections that is eventually below every other projection. A measure in $X$ can be given by:

$$\mu(B) := \lim_{n\to\omega} \mu_{e_n}(B \cap D_{e_n}), \quad B \subset X,$$

where $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ is a free ultrafilter. It is routine to show that $\mu$ is then a probability measure on $X$ satisfying the domain measurability and localization conditions mentioned above, i.e., $\mu(A) = \mu(sA)$ when $A \subset D_{s^*s}$ and $\mu(D_{s^*s}) = 1$ for every $s \in S$.

In order to prove (3) ⇒ (1) observe that the condition (3) ensures the existence of a domain Følner sequence $\{F_n\}_{n\in\mathbb{N}}$ that is eventually localized within every domain $D_{s^*s}$, i.e., for every $s \in S$ there is a number $N \in \mathbb{N}$ with $F_n \subset D_{s^*s}$ for all $n \geq N$. Consider then a free ultrafilter $\omega$ on $\mathbb{N}$ and the measure

$$\mu(B) := \lim_{n\to\omega} |B \cap F_n| / |F_n|.$$

It follows from Theorem 4.2.14 that $\mu$ is a domain measure, which, in addition, satisfies

$$\mu(D_{s^*s}) = \lim_{n\to\omega} |D_{s^*s} \cap F_n| / |F_n| = \lim_{n\to\omega} |F_n| / |F_n| = 1,$$

for all $s \in S$, i.e., the measure is localized.

We will only sketch the proof (1) ⇒ (3) since it is just a refinement of the same reasoning as in Theorem 4.2.14. Let $\mu$ be an invariant measure on $X$. The corresponding mean $m: \ell^\infty(X) \to \mathbb{C}$ (see Proposition 4.2.10) satisfies $m(p_{s^*s}) = 1$ for all $s \in S$. Then, any net $h_\lambda$ converging to $m$ in norm must also satisfy $\|h_\lambda(1-p_{s^*s})\|_1 \to 0$ for all $s \in S$. In particular, this must also be the case for the approximation $h$ appearing in Lemma 4.2.11. To get the desired Følner set, we have to cut $h$ so that its whole support is within $D_{s^*s}$ for all $s \in \mathcal{F}$. For this, consider a finite $\mathcal{F} = \{s_1, \ldots, s_k\}$ and define the function

$$g := \frac{h\, p_{s_1^* s_1 \ldots s_k^* s_k}}{\left\|h\, p_{s_1^* s_1 \ldots s_k^* s_k}\right\|_1} .$$

The function $g$ has norm 1, is positive and has finite support, which is contained in $D_{s^*s}$ for all $s \in \mathcal{F}$. Furthermore $\|h - g\|_1 \le \varepsilon$. Thus, by substituting $h$ by $g$ in the proof of Theorem 4.2.14 and following the same construction, we obtain a Følner set $F$ within the support of $g$, that is, a Følner set within the requirements of the theorem. □

Note that Theorem 4.2.18 is not constructive in the sense that if $X$ is not amenable, then one knows some domain $D_{e_0}$ has to be paradoxical, but Theorem 4.2.18 does not tell which element $e_0$ satisfies this condition. In the case $S$ has a minimal projection $e_0$ then one can improve the preceding theorem. To do this we first prove that minimal projections are central.

**Lemma 4.2.19.** *Let $S$ be a discrete and countable inverse semigroup with a minimal projection $e_0 \in E(S)$. Then $e_0$ commutes with every $s \in S$.*

*Proof.* Note that $e_0 = e_0\, s e_0 s^* = s^* e_0 s\, e_0$ by the minimality of $e_0$. Thus, we obtain

$$e_0\, s = e_0\, s e_0 s^*\, s = e_0\, s\, e_0 = s\, s^* e_0 s\, e_0 = s\, e_0,$$

where, for the first equality we have multiplied the identity $e_0 = e_0 s e_0 s^*$ from the right by $s$, and for the last equality we have multiplied the identity $e_0 = s^* e_0 s e_0$ from the left by $s$. □

In the language of Green's relations, Lemma 4.2.19 tells us that being $\mathcal{L}$-related with $e_0$ is equivalent to being $\mathcal{R}$-related with $e_0$, for whenever $s^*s = e_0$ one has

$$ss^* = ss^*ss^* = se_0s^* = ss^*e_0,$$

meaning that $ss^* \le e_0$. Whence $ss^* = e_0$ by the minimality of $e_0$, and $s\,\mathcal{R}\,e_0$. Thus the $\mathcal{D}$-class of $e_0$ within $S$ is a group, and, hence, contains exactly one $\mathcal{H}$-class. Moreover, this group $e_0Se_0 = D_{e_0}$ is canonically isomorphic to $G(S)$, the maximal homomorphic image of $S$ (see the definition of $G(S)$ in Theorem 4.1.1).

**Proposition 4.2.20.** *Let $S$ be a discrete, countable inverse semigroup with a minimal projection $e_0 \in E(S)$ and let $\alpha: S \to \mathcal{I}(X)$ be a representation with $D_{e_0}$ nonempty. Then the following conditions are equivalent:*

*(1) $X$ is $S$-amenable.*
*(2) $D_{e_0}$ is not paradoxical.*
*(3) $D_{e_0}$ is domain Følner.*

*Proof.* The implication (1) ⇒ (2) is a particular case of Theorem 4.2.18. For (2) ⇒ (3) note that it follows from Lemma 4.2.19 that $D_{e_0} \subset X$ is an invariant subset for the action. Indeed, for any $s \in S$

$$s\left(D_{e_0} \cap D_{s^*s}\right) = sD_{e_0} \subset D_{e_0}.$$

The implication (2) ⇒ (3) therefore follows from Theorem 4.2.14 by considering, if necessary, the induced action of $S$ on $D_{e_0}$. Finally, (3) ⇒ (1) can be proven in a similar fashion to that of (3) ⇒ (1) in Theorem 4.2.18. □

**Remark 4.2.21.** Observe that Proposition 4.2.20, in the case that $\alpha$ is the canonical Wagner-Preston representation $v: S \to \mathcal{I}(S)$, is just a rewriting of Theorem 4.1.1 for the case when $S$ has a minimal projection $e_0$.

We conclude the section providing a class of examples of inverse semigroups with a minimal projection. In particular, relate the following with Lemma 2.1.9 in the context of algebras.

**Proposition 4.2.22.** *Let $S$ be a countable and discrete inverse semigroup. Suppose $S$ satisfies the Følner condition but not the proper Følner condition. Then $S$ has a minimal projection.*

*Proof.* Following Theorem 3.1.1 there is an element $a \in S$ such that $|Sa| < \infty$. Suppose $Sa = \{s_1 a, \ldots, s_k a\}$. Then we claim $e := s_1^* s_1 \ldots s_k^* s_k a a^* \in S$ is a minimal projection. Indeed, for any other projection $f \in E(S)$ there is an $i$ such that $s_i a = fa$. In this case we have

$$a^* f a = (fa)^* fa = (s_i a)^* s_i a = a^* s_i^* s_i a.$$

Thus, multiplying by $a$ from the left, $aa^* fa = fa = aa^* s_i^* s_i a = s_i^* s_i a$. Therefore

$$f \geq faa^* = s_i^* s_i aa^* \geq e,$$

proving that $e$ is indeed minimal. □

Note that, in order to produce an example of an inverse semigroup $S$ that is Følner but not proper Følner, as in the hypothesis of Proposition 4.2.22, $S$ must have a minimal projection $e_0 \in E(S)$. Moreover, by Proposition 4.2.22, the domain $D_{e_0}$ must be domain-Følner. For instance, it is straightforward to show that $S := \mathbb{F}_2 \sqcup \{0\}$ satisfies the Følner condition but not the proper one.

## 4.3 The uniform Roe algebra and 2-norm approximations

This part of the thesis connects the analysis of the previous section with properties of a C*-algebra $\mathcal{R}_X$, which generalizes the uniform Roe algebra $\ell^\infty(G) \rtimes_r G$ of a group $G$. In particular, we will show that domain measurability completely characterizes the existence of traces in $\mathcal{R}_X$. For now, let us introduce these algebras.

Let $S$ be a discrete inverse semigroup and consider a representation $\alpha\colon S \to \mathcal{I}(X)$ on a discrete set $X$. As before, we will denote $\alpha_s(x)$ simply by $sx$ for any $s \in S$ and $x \in D_{s^*s}$. To construct the C*-algebra $\mathcal{R}_X$ consider first the representation $V\colon S \to \mathcal{B}(\ell^2 X)$ given by

$$V_s\delta_x := \begin{cases} \delta_{sx} & \text{if } x \in D_{s^*s} \\ 0 & \text{otherwise,} \end{cases}$$

where $\{\delta_x\}_{x\in X} \subset \ell^2 X$ is the canonical orthonormal basis. $V$ is a $*$-representation of $S$ in terms of partial isometries of $\ell^2 X$ and, moreover, in case that $\alpha$ is the canonical Wagner-Preston representation $v\colon S \to \mathcal{I}(S)$ then $V$ is usually called the *left regular representation of* $S$. Define the unital *-subalgebra $\mathcal{R}_{X,alg}$ in $\mathcal{B}(\ell^2 X)$ as the algebra generated by monomials of the form $fV_s$, where $s \in S$ and $f \in \ell^\infty X$. The C*-algebra $\mathcal{R}_X$ is then defined as the norm closure of $\mathcal{R}_{X,alg}$, i.e.,

$$\mathcal{R}_X := \overline{\mathcal{R}_{X,alg}} = \mathrm{C}^*\Big(\ell^\infty(X) \cdot \{V_s \mid s \in S\}\Big) \subset \mathcal{B}\left(\ell^2 X\right).$$

Of surprising (and subtle) importance is the fact that $\mathcal{R}_X$ may not be unital. Indeed, if $S$ does not contain a unit element then $\mathcal{R}_X$ above may not be unital itself since, *a priori*, $\ell^\infty(X)$ may not be contained in $\mathcal{R}_X$. For instance, if $S = \mathbb{N}$ as a set, and $n \cdot m := \min\{n, m\}$, then it is not hard to show that $\mathcal{R}_X = c_0(\mathbb{N})$, which is a non-unital C*-algebra (see also Example 5.2.26). This subtle behavior, however, will not be relevant in this chapter but in the following, where we will see when $\mathcal{R}_X$ is actually a uniform Roe algebra (see Section 5.2.2).

Note that conjugation by $V_s$ implements the action of $s \in S$ on subsets $A \subset X$. Indeed, it is straightforward to show the following intertwining equation for the generators of $\mathcal{R}_X$:

$$p_A V_s = V_s p_{s^*(A \cap D_{ss^*})}, \quad \text{where } s \in S \text{ and } A \subset X,$$

where, as before, $p_A \in \mathcal{B}(\ell^2 X)$ is the orthogonal projection onto the closure of $\operatorname{span}\{\delta_a \mid a \in A\}$. Note also that for any $s \in S$, $f \in \ell^\infty(X)$ (which we interpret as multiplication operators on $\ell^2 X$) we have the following commutation relations between the generators of $\mathcal{R}_X$:

$$V_s f = \big(sf\big) V_s \quad \text{or, equivalently,} \quad f V_s = V_s \big(s^* f\big).$$

### 4.3.1 A crossed product description of $\mathcal{R}_X$ when $X = S$

The goal of this brief section is to give an alternative description of the algebra $\mathcal{R}_X$ introduced above in the particular case that $X = S$ and $S$ acts on $S$ via the usual Wagner-Preston representation $v$ (see Proposition 3.3.4). Observe that if $S = G$ happens to be a group then the algebra $\mathcal{R}_G$ is usually referred to as the *uniform Roe algebra* of the group, and one can prove that it is isomorphic to the crossed product $\ell^\infty(G) \rtimes_r G$, where the action of $G$ on $\ell^\infty(G)$ is the usual shift in the argument (see [20, Proposition 5.1.3]). This construction was generalized to the setting of inverse semigroups (see, for instance, [101, 36, 68]), and since in the subsequent sections we will only deal with the commutative case $A = \ell^\infty(S)$, that is the only case we introduce.

Consider the action of $S$ on $\ell^\infty(S)$ introduced in Eq. (4.1). In order to construct the *reduced crossed product of $\ell^\infty(X)$ by $S$* recall that the canonical representation of $\ell^\infty(X)$ as multiplication operators in $\ell^2 X$ is faithful, and consider

$$\pi\colon \ell^\infty(S) \to \mathcal{B}\left(\ell^2 S \otimes \ell^2 S\right), \quad \left(\pi(f)\right)(\delta_x \otimes \delta_y) := \begin{cases} f(yx)\,\delta_x \otimes \delta_y & \text{if } x \in D_{y^*y} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$1 \otimes V\colon S \to \mathcal{B}\left(\ell^2 S \otimes \ell^2 S\right), \quad (1 \otimes V_s)(\delta_x \otimes \delta_y) := \begin{cases} \delta_x \otimes \delta_{sy} & \text{if } y \in D_{s^*s} \\ 0 & \text{otherwise,} \end{cases}$$

where $\{\delta_x\}_{x \in X}$ denotes the canonical orthogonal basis of $\ell^2 S$. Observe the representations intertwine the action in the following covariant way

$$(1 \otimes V_s)\,\pi(f)\,(1 \otimes V_s)^* = \pi(sf)$$

for all $s \in S$ and $f \in E_{s^*s}$. The *reduced crossed product* $\ell^\infty(S) \rtimes_r S$ is then the C*-algebra generated by the images of $\pi$ and $1 \otimes V$, that is:

$$\ell^\infty(S) \rtimes_r S := C^*\Big(\pi\left(\ell^\infty(S)\right) \cdot \{1 \otimes V_s \mid s \in S\}\Big) \subset \mathcal{B}\left(\ell^2 S \otimes \ell^2 S\right).$$

The following result relates this construction to the algebra $\mathcal{R}_S$, and generalizes a standard result for groups (see [20, Proposition 5.1.3]).

**Theorem 4.3.1.** *Let $S$ be a countable and discrete inverse semigroup and consider the action of $S$ on $\ell^\infty(S)$ defined in Proposition 4.2.9. Then $\mathcal{R}_S \cong \ell^\infty(S) \rtimes_r S$.*

*Proof.* Consider the bounded linear operator $U\colon \ell^2 S \otimes \ell^2 S \to \ell^2 S \otimes \ell^2 S$ given by

$$U(\delta_x \otimes \delta_y) = \begin{cases} \delta_x \otimes \delta_{yx} & \text{if } xx^* = y^*y \\ 0 & \text{otherwise.} \end{cases}$$

It can be checked that $U$ is a partial isometry whose adjoint is

$$U^*(\delta_u \otimes \delta_v) = \begin{cases} \delta_u \otimes \delta_{vu^*} & \text{if} \quad u^*u = v^*v \\ 0 & \qquad \text{otherwise.} \end{cases}$$

Moreover, the initial projection $U^*U$ is the orthogonal projection onto the closure of the subspace generated by $\delta_x \otimes \delta_y$ where $xx^* = y^*y$, and the final projection $UU^*$ projects onto the subspace generated by $\delta_u \otimes \delta_v$ where $u^*u = v^*v$. In addition, it is routine to check that

$$U\pi(f)U^* = (1 \otimes f)UU^* = UU^*(1 \otimes f) \text{ and } U(1 \otimes V_s) = (1 \otimes V_s)U.$$

It follows that the map $\mathrm{Ad}(U)$ restricts to an *-isomorphism between $\ell^\infty(S) \rtimes_r S$ and $(1 \otimes \mathcal{R}_S) \subset 1 \otimes \mathcal{B}(\ell^2 S)$. Indeed, from the commutation relations above we have that

$$\begin{aligned} U\big(\ell^\infty(S) \rtimes_r S\big)U^* &= \mathrm{cl}_{\|\cdot\|}\Big(\mathrm{span}\Big\{1 \otimes fV_s \mid f \in \ell^\infty(S) \text{ and } s \in S\Big\}\Big) \cdot UU^* \\ &= \big(1 \otimes \mathcal{R}_S\big)UU^*, \end{aligned}$$

which, in turn, is $*$-isomorphic to $1 \otimes \mathcal{R}_S$. □

Observe that, in the more general case when $S$ is represented in some $\mathcal{I}(X)$, then one could try to use the same argument in order to prove that $\ell^\infty(X) \rtimes_r S \cong \mathcal{R}_X \subset \mathcal{B}(\ell^2 X)$. The same argument, however, does not work. Indeed, the proof above systematically uses that $x, y, u, v \in S$ and, therefore, operations of the form $x^*x, yx$ and $uv^*$ make sense. In the case $S \neq X$ these operations are not well defined, since $x, y, u, v$ are points in $X$ and, therefore, one would need to change the method of proof.

### 4.3.2 Traces vs. proper infiniteness

After the detour of Section 4.3.1, we come back to some amenability notions, which are the main topic of this chapter. Recall that we aim to give alternative characterizations to domain-measurability by expanding Theorem 4.2.14 to the setting of C*-algebras. In order to do this we first need to give some notation and some preliminary lemmas.

Let $\alpha: S \to \mathcal{I}(X)$ be a representation of an inverse semigroup $S$ on a discrete set $X$. To avoid technical complications suppose that $X = \cup_{e \in E(S)} D_e$.[1] Given a pair $x, y \in X$, we write $x\mathcal{L}y$ if there is some $s \in S$ such that $x \in D_{s^*s}$ and $sx = y$. Observe that the relation $\mathcal{L}$ is an equivalence relation. Indeed, symmetry follows from the fact that if $x\mathcal{L}y$ then $y\mathcal{L}x$ by considering $y = sx \in D_{ss^*}$. For transitivity, if $x\mathcal{L}y\mathcal{L}z$ then $x \in D_{s^*t^*ts}$, where $s, t$ witness $x\mathcal{L}y$ and $y\mathcal{L}z$ respectively. Lastly, observe that, since $x \in X = \cup_{e \in E(S)} D_e$, then there is some projection $e \in E(S)$ such that $x \in D_e$.

[1]This is usually known as *non-degeneracy of the representation.*

Since $e$ is an idempotent it then follows that $ex = x$ and, hence, $x\mathcal{L}x$ for every $x \in X$, proving reflexivity.

**Remark 4.3.2.** Observe that the equivalence relation $\mathcal{L}$ implies that a point $x$ can be moved onto $y$ via the action of the inverse semigroup $S$. Moreover, we use the letter $\mathcal{L}$ to denote the equivalence relation since it generalizes Green's $\mathcal{L}$-relation when the representation $\alpha: S \to \mathcal{I}(X)$ is actually equal to the usual Wagner-Preston representation $v: S \to \mathcal{I}(S)$. Indeed, note the following straightforward fact.

**Lemma 4.3.3.** *Let $S$ be an inverse semigroup and let $s \in S$. Then $x \in D_{s^*s}$ if and only if $x\mathcal{L}sx$.*

*Proof.* Assume $x\mathcal{L}sx$. Then $x^*x = x^*s^*sx$ and, hence, multiplying by $x$ from the left we obtain $x = xx^*x = xx^*s^*sx = s^*sx$ which shows $x \in D_{s^*s}$. The reverse is done similarly. □

In particular, if $\alpha = v$ is the usual Wagner-Preston representation, then there is some $s \in S$ taking $x$ to $y$ (i.e., $x \in D_{s^*s}$ and $sx = y$) if, and only if, $x\mathcal{L}y$. The latter Lemma 4.3.3 will be used throughout the rest of the thesis, mainly in Chapter 5, almost always without mention.

Finally, observe that, from our perspective, the current use of $\mathcal{L}$ is that if a set $F \subset X$ has only one $\mathcal{L}$-class, then the corner $p_F \mathcal{R}_X p_F$ has dimension $|F|^2$ as a vector space (see Lemma 4.3.5 below). Moreover, as the following lemma proves, we may always assume that domain-Følner sets have exactly one $\mathcal{L}$-class.

**Lemma 4.3.4.** *Let $\alpha: S \to \mathcal{I}(X)$ be a representation of $S$ on $X$. If $A \subset X$ is domain Følner then for every $\varepsilon > 0$ and finite $\mathcal{F} \subset S$ there is an $(\varepsilon, \mathcal{F})$-domain Følner $F_0 \subset A$ with exactly one $\mathcal{L}$-class.*

*Proof.* Since $A$ is domain Følner for any $\varepsilon > 0$ and finite $\mathcal{F} \subset S$ there is a finite $F \subset A$ such that
$$|s(F \cap D_{s^*s}) \setminus F| < \frac{\varepsilon}{|\mathcal{F}|} |F| \quad \text{for all } s \in \mathcal{F}.$$
Decomposing $F$ into its $\mathcal{L}$-classes we get $F = F_1 \sqcup \cdots \sqcup F_\ell$, where $u\mathcal{L}v$ if and only if $u, v \in F_i$ for some $i$. To prove the claim it is enough to prove that some $F_j$ must be $(\varepsilon, \mathcal{F})$-domain Følner. Indeed, suppose none were, that is, for all $j = 1, \ldots, \ell$ there is an $s_j \in \mathcal{F}$ such that
$$\left| s_j \left( F_j \cap D_{s_j^* s_j} \right) \setminus F_j \right| \geq \varepsilon |F_j|.$$
Observe that the choice of $s_j$ is not unique, but we can consider a particular fixed choice. Arrange then the indices according to the following: for $s \in \mathcal{F}$ consider $\Lambda_s := \{j \in \{1, \ldots, \ell\} \mid s_j = s\}$. Note that some $\Lambda_s$ might be empty. Define $F_s := \sqcup_{i \in \Lambda_s} F_i$ and observe
$$|s(F_s \cap D_{s^*s}) \setminus F_s| = \sum_{j \in \Lambda_s} \left| s_j \left( F_j \cap D_{s_j^* s_j} \right) \setminus F_j \right| \geq \varepsilon \sum_{j \in \Lambda_s} |F_j| = \varepsilon |F_s|.$$

Taking the sum over all $s \in \mathcal{F}$ we get

$$\begin{aligned}\varepsilon |F| > \sum_{s\in\mathcal{F}} |s\,(F \cap D_{s^*s}) \smallsetminus F| &= \sum_{s,t\in\mathcal{F}} |s\,(F_t \cap D_{s^*s}) \smallsetminus F_t| \\ &\geq \sum_{s\in\mathcal{F}} |s\,(F_s \cap D_{s^*s}) \smallsetminus F_s| \geq \varepsilon \sum_{s\in\mathcal{F}} |F_s| = \varepsilon\,|F|\,.\end{aligned}$$

This is a contradiction and, thus, some $F_{j_0}$ must witness the domain Følner condition. □

**Lemma 4.3.5.** *Let $\alpha\colon S \to \mathcal{I}(X)$ be a representation of $S$ on $X$. Let $F_1, F_2 \subset X$ be finite sets such that $F_1 \cup F_2$ has only one $\mathcal{L}$-class. Then $W := p_{F_2}\mathcal{R}_X p_{F_1}$ has linear dimension $|F_1||F_2|$.*

*Proof.* To prove the claim it suffices to show that for every $u_i \in F_i, i = 1,2$ the matrix unit

$$M_{u_2,u_1}\delta_x = \begin{cases} \delta_{u_2} & \text{if } x = u_1 \\ 0 & \text{otherwise,} \end{cases}$$

is contained in $W$. Since $u_1\mathcal{L}u_2$ there is an element $s \in S$ such that $u_1 \in D_{s^*s}$ and $su_1 = u_2$. Therefore

$$M_{u_2,u_1} = p_{F_2}p_{\{u_2\}}V_s p_{\{u_1\}}p_{F_1} \in p_{F_2}\mathcal{R}_X p_{F_1} = W,$$

and hence $\dim W = |F_1||F_2|$. □

The following result defines a canonical conditional expectation from $\mathcal{B}(\ell^2X)$ onto $\ell^\infty(X)$, and its proof is routine.

**Proposition 4.3.6.** *The linear map $\mathcal{E}\colon\mathcal{B}(\ell^2X) \to \ell^\infty(X)$ given by*

$$\mathcal{E}\,(T) = \sum_{x\in X} p_{\{x\}}Tp_{\{x\}},$$

*is a conditional expectation, where the sum is taken in the strong operator topology.*

We are now in a position to show the main theorem of the section, which characterizes the domain measurability of the action in terms of amenable traces of the algebra $\mathcal{R}_X$.

**Theorem 4.3.7.** *Let $S$ be a countable and discrete inverse semigroup with identity $1 \in S$, and let $\alpha\colon S \to \mathcal{I}(X)$ be a representation of $S$ on $X$. Then the following are equivalent:*

*(1) $X$ is $S$-domain measurable.*
*(2) $X$ is not $S$-paradoxical.*
*(3) $X$ is $S$-domain Følner.*
*(4) $\mathcal{R}_{X,alg}$ is algebraically amenable.*

*(5) $\mathcal{R}_X$ has an amenable trace (and hence is a Følner $C^*$-algebra).*

*(6) $\mathcal{R}_X$ is not properly infinite.*

*(7) $[0] \neq [1]$ in the $K_0$-group of $\mathcal{R}_X$.*

*Proof.* The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Theorem 4.2.14.

(1) $\Rightarrow$ (5). Consider the conditional expectation $\mathcal{E}: \mathcal{B}(\ell^2 X) \to \ell^\infty(X)$ introduced in Proposition 4.3.6. Since $X$ is $S$-domain measurable, by Proposition 4.2.10, there is a mean $m: \ell^\infty(X) \to \mathbb{C}$ satisfying $m(sf) = m(f)$ for every $f \in E_{s^*s}$. We claim that then $\phi(T) := m(\mathcal{E}(T))$ is a hypertrace on $\mathcal{R}_X$. Indeed, observe that linearity, positivity and normalization follow from those of $m$ and $\mathcal{E}$. Hence we only have to prove the hypertrace property for the generators of $\mathcal{R}_X$. Note that since $\mathcal{E}$ is a conditional expectation we have $\phi(fT) = \phi(Tf)$ for any $f \in \ell^\infty(X)$, $T \in \mathcal{B}(\ell^2 X)$. To show the same relation for the generator $V_s$ note first that for any $s, t \in S$ the following relation holds:

$$\mathcal{E}(V_s T)(x) = \begin{cases} T_{s^*x,x} & \text{if } x \in D_{ss^*}, \\ 0 & \text{if } x \notin D_{ss^*}, \end{cases} \quad \text{and} \quad \mathcal{E}(TV_s)(y) = \begin{cases} T_{y,sy} & \text{if } y \in D_{s^*s}, \\ 0 & \text{if } y \notin D_{s^*s}. \end{cases}$$

It follows from the action introduced in Eq. (4.1) that $s\mathcal{E}(TV_s) = \mathcal{E}(V_s T)$ and $\mathcal{E}(TV_s) = \mathcal{E}(TV_s)V_{s^*s}$ and thus

$$\begin{aligned} \phi(V_s T) &= m(\mathcal{E}(V_s T)) = m(s \cdot \mathcal{E}(TV_s)) \\ &= m(\mathcal{E}(TV_s)V_{s^*s}) = m(\mathcal{E}(TV_s)) = \phi(TV_s), \end{aligned}$$

where we used the invariance of the mean in the third equality.

(5) $\Rightarrow$ (6). Suppose that $\mathcal{R}_X$ is properly infinite and has a hypertrace $\phi$. Then we obtain a contradiction from

$$1 = \phi(1) \geq \phi(W_1 W_1^*) + \phi(W_2 W_2^*) = \phi(W_1^* W_1) + \phi(W_2^* W_2) = \phi(1) + \phi(1) = 2,$$

where $W_1, W_2$ are the isometries witnessing the proper infiniteness of $1 \in \mathcal{R}_X$.

(6) $\Rightarrow$ (2). Suppose that $s_i, t_j, A_i, B_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, implement the $S$-paradoxicality of $X$, that is, $A_i \subseteq D_{s_i^* s_i}$, $B_j \subseteq D_{t_j^* t_j}$, and

$$\begin{aligned} X &= s_1 A_1 \sqcup \ldots \sqcup s_n A_n = t_1 B_1 \sqcup \ldots \sqcup t_m B_m \\ &\supset A_1 \sqcup \ldots \sqcup A_n \sqcup B_1 \sqcup \ldots \sqcup B_m. \end{aligned}$$

Consider now the operators

$$W_1 := V_{s_1^*} p_{s_1 A_1} + \cdots + V_{s_n^*} p_{s_n A_n} \quad \text{and} \quad W_2 := V_{t_1^*} p_{t_1 B_1} + \cdots + V_{t_m^*} p_{t_m B_m}.$$

These are both partial isometries, since $V_{s_i^*} p_{s_i A_i}$ and $V_{t_j^*} p_{t_j B_j}$ are partial isometries with pairwise orthogonal domain and range projections. Furthermore, $W_1^* W_1$ is the projection onto the union of the domains of $V_{s_i^*} p_{s_i A_i}$, which is the whole space $\ell^2 X$. The same argument proves that $W_2^* W_2 = 1$. Therefore, to prove the claim we just have to show that $W_1 W_1^*$ and $W_2 W_2^*$ are orthogonal projections. But these correspond to projections onto $\sqcup_{i=1}^n A_i$ and $\sqcup_{j=1}^m B_j$, respectively, which are disjoint sets in $X$.

(3) $\Rightarrow$ (4). By Lemma 4.3.4, given $\varepsilon > 0$ and finite $\mathcal{F} \subset S$, there is a finite non-empty $F \subset X$ with exactly one $\mathcal{L}$-class such that

$$|s(F \cap D_{s^*s}) \setminus F| < \varepsilon |F|, \quad \text{for all } s \in \mathcal{F}.$$

Consider the space $W := p_F \mathcal{R}_X p_F = p_F \mathcal{R}_{X,alg} p_F$ and observe

$$\begin{aligned} V_s W &= \{V_s p_F T p_F \mid T \in \mathcal{R}_{X,alg}\} \\ &= \left\{p_{s(F\cap D_{s^*s})} V_s T p_F \mid T \in \mathcal{R}_{X,alg}\right\} \\ &\subset \left\{p_{s(F\cap D_{s^*s})\setminus F} V_s T p_F \mid T \in \mathcal{R}_{X,alg}\right\} + \left\{p_F p_{s(F\cap D_{s^*s})} V_s T p_F \mid T \in \mathcal{R}_{X,alg}\right\}. \end{aligned}$$

Therefore

$$V_s W + W \subset p_{s(F\cap D_{s^*s})\setminus F} \cdot \left(\mathcal{R}_{X,alg}\right) \cdot p_F + p_F \left(\mathcal{R}_{X,alg}\right) p_F \,,$$

and, by Lemma 4.3.5,

$$\dim(W + V_s W) \le |F|^2 + |F| \, |s(F \cap D_{s^*s}) \setminus F| \le (1+\varepsilon) \dim(W),$$

which proves the algebraic amenability of $\mathcal{R}_{X,alg}$.

(4) $\Rightarrow$ (5). This follows from one of the main results of [8], which states that if a pre-C*-algebra is algebraically amenable then its closure has an amenable trace (see [8, Theorem 3.17]).

(5) $\Rightarrow$ (7). Any trace $\phi$ on $\mathcal{R}_X$, by the universal property of the $K_0$ group, induces a group homomorphism $\phi_0 : K_0(\mathcal{R}_X) \to \mathbb{R}$ such that $\phi_0([p]) = \phi(p)$ for any projection $p \in \mathcal{R}_X$. In this case $\phi_0([1]) = \phi(1) = 1$ while $0 = \phi(0) = \phi_0([0])$, and hence $[1] \neq [0]$.

(7) $\Rightarrow$ (2). If $X$ is $S$-paradoxical, then it follows from Lemma 4.2.7 and the same argument as in the implication (6) $\Rightarrow$ (2) that $[1] = [1] + [1]$ in $K_0(\mathcal{R}_X)$. Therefore $[1] = [0]$ in $K_0(\mathcal{R}_X)$. ☐

Theorem 4.3.7 can be *somewhat generalized* to hold for any set $A \subset X$.

**Corollary 4.3.8.** *Let $S$ be a countable and discrete inverse semigroup with identity $1 \in S$, and let $\alpha : S \to \mathcal{I}(X)$ be a representation of $S$ on $X$. Let $A \subset X$, then the following conditions are equivalent:*

*(1) $A$ is $S$-domain measurable.*

*(2) $A$ is not $S$-paradoxical.*

*(3) There is a tracial weight $\psi : \mathcal{R}_X^+ \to [0, \infty]$ such that $\psi(p_A) = 1$.*

*(4) $p_A \in \mathcal{R}_X$ is not a properly infinite projection.*

*Proof.* Most of the proof is similar as in the reasoning of Theorem 4.3.7. The only difference regards condition (3), and the replacement of the trace with a weight. To prove that (3) $\Rightarrow$ (1), consider the measure $\mu(B) = \psi(p_B)$. This $\mu$ will then be a measure on $X$ such that $\mu(A) = 1$. Furthermore, invariance follows from $\psi$ being tracial:

$$\mu(B) = \psi(V_{s^*s}p_B) = \psi(V_s p_B V_{s^*}) = \psi(p_{sB}) = \mu(\alpha_s(B)) \tag{4.6}$$

for every $B \subset D_{s^*s}$.

To prove (1) $\Rightarrow$ (3) we adapt the ideas in [91, Proposition 5.5]. Given a domain measure $\mu$ normalized at $A$, denote by $\mathcal{P}_{fin}(X)$ the (upwards directed) set of $K \subset X$ with finite measure, i.e., $\mu(K) < \infty$. Given $K \in \mathcal{P}_{fin}(X)$ consider the finite measure $\mu_K(B) = \mu(K \cap B)$ and extend it, as in Proposition 4.2.10, to a functional $m_K$. Given a non-negative $f \in \ell^\infty(X)$ define

$$m(f) := \sup_{K \in \mathcal{P}_{fin}(X)} m_K(f).$$

Then $m$ is $\mathbb{R}_+$-linear, lower-semicontinuous, normalized at $p_A$, and satisfies that $m(sf) = m(s^*sf)$, for every $s \in S, f \in \ell^\infty(X)$. Finally, the weight given by $\psi := m \circ \mathcal{E}: \mathcal{R}_X^+ \to [0, \infty]$ is a tracial weight. □

In our last result of the section we point out that one can translate the Følner sequences of $X$ onto Følner sequences of projections in $\mathcal{R}_X$. To prove it we first recall the following known result, which states that Følner sequences are preserved under C*-closure (see [14, Section 1]). Its proof, in addition, is straightforward.

**Lemma 4.3.9.** *Let $\mathcal{T} \subset \mathcal{B}(\mathcal{H})$ be a set of operators on a separable Hilbert space $\mathcal{H}$. Suppose $\mathcal{T}$ has a Følner sequence $\{p_n\}_{n=1}^\infty$. Then $\{p_n\}_{n=1}^\infty$ is a Følner sequence for the C*-algebra generated by $\mathcal{T}$.*

**Proposition 4.3.10.** *Let $S$ be a countable and discrete inverse semigroup with identity, and let $\alpha: S \to \mathcal{I}(X)$ be a representation. If $X$ is domain measurable, then there is a sequence $\{p_n\}_{n=1}^\infty \subset \mathcal{R}_X$ which is a Følner sequence of projections for each operator $T \in \mathcal{R}_X$.*

*Proof.* Choose a $S$-domain Følner sequence $\{F_n\}_{n=1}^\infty$ of $X$, that exists since $X$ is domain measurable (see Theorem 4.3.7). Consider the orthogonal projection $p_n$ onto $\operatorname{span}\{\delta_f \mid f \in F_n\} \subset \ell^2 X$. Clearly, $p_n$ lies within $\mathcal{R}_X$, so, by Lemma 4.3.9, it is enough to show that it is a Følner sequence for all generating elements $V_s p_A$, $s \in S$, $A \subset X$. For this we compute

$$(V_s p_A p_n)(\delta_x) = \begin{cases} \delta_{sx} & \text{if } x \in D_{s^*s} \cap F_n \cap A \\ 0 & \text{otherwise,} \end{cases}$$

$$(p_n V_s p_A)(\delta_x) = \begin{cases} \delta_{sx} & \text{if } x \in A \cap s^*(F_n \cap D_{ss^*}) \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have the following estimates in the Hilbert-Schmidt norm:

$$\begin{aligned}\|V_s p_A p_n - p_n V_s p_A\|_2^2 &\leq |\{x \in F_n \cap D_{s^*s} \mid sx \notin F_n\}| + |s^*(F_n \cap D_{ss^*}) \smallsetminus F_n| \\ &= |s(F_n \cap D_{s^*s}) \smallsetminus F_n| + |s^*(F_n \cap D_{ss^*}) \smallsetminus F_n|\,.\end{aligned}$$

Noting that $\|p_n\|_2^2 = |F_n|$ the result follows from normalizing by $|F_n|$ and taking limits on both sides of the inequality. □

### 4.3.3 (Amenable) Traces as domain-measures

Continuing the research conducted so far we now prove that either all traces in $\mathcal{R}_X$ are amenable or this algebra has no traces at all. Observe that similar results hold for nuclear C*-algebras (see [20, Proposition 6.3.4]), for uniform Roe algebras over metric spaces (see [8, Corollary 4.15]) and for C*-algebras with the *weak expectation property* (see [19, Proposition 4.4.2]).

The main technical tool of the section is the following lemma, which states that traces of $\mathcal{R}_X$ factor through $\ell^\infty(X)$ via the conditional expectation $\mathcal{E}$ (see Proposition 4.3.6).

**Lemma 4.3.11.** *Let $\alpha\colon S \to \mathcal{I}(X)$ be a representation and let $\phi\colon \mathcal{R}_X \to \mathbb{C}$ be a trace. Then $\phi(T) = \phi(\mathcal{E}(T))$ for every $T \in \mathcal{R}_X$, where $\mathcal{E}$ denotes the canonical conditional expectation onto $\ell^\infty(X)$.*

*Proof.* Since the closure of the linear span of the elements of the form $V_s p_A$, where $s \in S$ and $A \subset D_{s^*s}$, is dense in $\mathcal{R}_X$ it is enough to show the claim for these elements.

First suppose that $A$ has no fixed points under $s$, i.e., $\{a \in A \mid sa = a\} = \varnothing$. Consider the graph whose vertices are the elements of $A$ and such that two vertices $a, b$ are joined by an edge if and only if $b = sa$. Since the action of $s$ is injective on $A$ it is clear that every vertex has at most degree 2 and no loops. Therefore it can be colored by 3 colors and, thus, there is a partition $A = B_1 \sqcup B_2 \sqcup B_3$ such that if $a \in B_i$ then $sa \notin B_i$. This allows us to decompose $V_s p_A$ as

$$V_s p_A = V_s p_{B_1} + V_s p_{B_2} + V_s p_{B_3}.$$

Taking traces on each side of the equality gives

$$\phi(V_s p_A) = \phi(p_{B_1} V_s p_{B_1}) + \phi(p_{B_2} V_s p_{B_2}) + \phi(p_{B_3} V_s p_{B_3}) = 0\,.$$

But in this case we also have $\mathcal{E}(V_s p_A) = 0$ and the equality $\phi = m \circ \mathcal{E}$ follows.

Secondly, for arbitrary $s \in S$ and $A \subset D_{s^*s}$ one can decompose $A = B \sqcup C$, where $B := \{a \in A \mid sa = a\}$ is the set of fixed points and $C := \{a \in A \mid sa \neq a\}$. In this case

$$V_s p_A = V_s p_B + V_s p_C.$$

By the above paragraph it follows that $\phi(V_s p_C) = 0$, while $V_s p_B$ is a projection in $\ell^\infty(X)$. Thus

$$\phi(V_s p_A) = \phi(V_s p_B) + \phi(V_s p_C) = \phi(\mathcal{E}(V_s p_B)) = \phi(\mathcal{E}(V_s p_A)),$$

which concludes the proof. □

The following theorem is a consequence of Theorem 4.3.7 and the preceding lemma.

**Theorem 4.3.12.** *Let $S$ be a countable and discrete inverse semigroup with identity $1 \in S$, and let $\alpha: S \to \mathcal{I}(X)$ be a representation. Consider a positive linear functional $\phi$ on $\mathcal{R}_X$. Then the following conditions are equivalent:*

*(1) $\phi$ is an amenable trace on $\mathcal{R}_X$.*
*(2) $\phi$ is a trace on $\mathcal{R}_X$.*
*(3) $\phi = \phi|_{\ell^\infty(X)} \circ \mathcal{E}$ and the measure $\mu(A) := \phi(p_A)$ satisfies domain measurability, i.e., $\mu(A) = \mu(sA)$ for all $s \in S$ and $A \subset D_{s^*s}$.*

*Proof.* The fact that (1) ⇒ (2) is obvious. For (2) ⇒ (3) note that by Lemma 4.3.11 it remains only to prove that $\mu(A) = \mu(sA)$ for every $s \in S$, $A \subset D_{s^*s}$. This follows from $\phi$ being a trace and an argument similar to Eq. (4.6). Finally, (3) ⇒ (1) is proved as the implication (1) ⇒ (5) in Theorem 4.3.7. □

Having done all the *heavy-lifting*, we end the section collecting some consequences of the theorems so far presented. The following results easily follow from theorems before.

The first corollary acknowledges the fact that there is a canonical bijection between the set of domain-measures of $X$ and the set of traces of $X$, all of which are necessarily amenable.

**Corollary 4.3.13.** *Let $S$ be a countable and discrete inverse semigroup with identity $1 \in S$, and let $\alpha: S \to \mathcal{I}(X)$ be a representation. Then:*

*(1) $X$ is $S$-domain measurable if and only if there is a trace on $\mathcal{R}_X$.*
*(2) Every trace on $\mathcal{R}_X$ is amenable.*
*(3) There is a canonical bijection between the space of measures on $X$ such that $\mu(A) = \mu(sA)$ when $s \in S$ and $A \subset D_{s^*s}$ and the space of traces of $\mathcal{R}_X$.*

Lastly, the following, which we state as *theorem* in analogy to Theorem 4.2.18, characterizes the amenable actions in C*-terms, as in Theorem 4.3.7.

**Theorem 4.3.14.** *Let $S$ be a countable and discrete inverse semigroup with identity $1 \in S$ and let $\alpha: S \to \mathcal{I}(X)$ be a representation. Then the following conditions are equivalent:*

*(1) $X$ is $S$-amenable.*

*(2) $D_e$ is not $S$-paradoxical for any $e \in E(S)$.*
*(3) $\mathcal{R}_X$ has a trace $\phi$ such that $\phi(V_e) = 1$ for all $e \in E(S)$.*
*(4) No projection $V_e \in \mathcal{R}_X$ is properly infinite.*

## 4.4 Examples

This section of the chapter aims to give some examples of the notions and constructions dealt with in the previous sections. We do so with the point of view of operator algebras, that is, we are going to introduce some customary constructions that come, mainly, from the area of C*-algebras. Of course, the following subsections do not intend to be a complete list of interesting examples and, moreover, there might be significant overlap between them. Last, and most certainly not least, we warn the reader we will use a more colloquial language in this section, mainly due to the fact that most of the constructions are well known.

To start off, we remark the following result. Given an inverse semigroup $S$ we say $S$ is *locally finite* if every finite set $K \subset S$ generates a finite subsemigroup, i.e., $T := \langle K \rangle$ is finite. There are multiple instances of locally finite semigroups but, from our perspective, locally finite semigroups are always domain-measurable.

**Proposition 4.4.1.** *Let $S$ be a locally finite inverse semigroup, and let $\alpha\colon S \to \mathcal{I}(X)$ be a representation. Then $X$ is $S$-domain-measurable. Moreover, if $D_{s^*s}$ is non-empty for all $s \in S$ then $X$ is $S$-amenable.*

*Proof.* Given a finite $\mathcal{F} \subset S$ let $T := \langle \mathcal{F} \rangle$. Fix a point $x_0 \in X$ and consider the $T$-orbit of $x_0$:

$$F := \{tx_0 \mid x_0 \in D_{t^*t} \text{ and } t \in T\}.$$

By construction $F$ is invariant under the action of $T$, that is, it is a domain-Følner set of the representation. By Theorem 4.2.14 this proves the first claim.

For the last statement, note that if $D_{s^*s}$ is non-empty for all $s \in S$ then $\cap_{t \in T} D_{t^*t}$ is non-empty as well, and taking any $x_0 \in \cap_{t \in T} D_{t^*t}$ and following the same procedure yields a Følner set $F$ within the set $\cap_{t \in T} D_{t^*t}$. Therefore $X$ is amenable by Theorem 4.2.18. □

The following remark is in order.

**Remark 4.4.2.** Observe that, in Proposition 4.4.1, we can only guarantee that $X$ is domain measurable unless $D_{s^*s}$ is non-empty for all $s \in S$. Indeed, let $B_2 = \{e, f, s, s^*, 0\}$ be the semigroup with multiplication table

$$e = s^*s, \quad f = ss^* \quad \text{and} \quad s^2 = 0.$$

Observe that $B_2$ is the set of matrix units of size 2 with a zero element adjoined (see [58, Proposition 6] for the relevance of $B_2$ in related areas). Moreover, $B_2$ can be represented on $X = \{a, b\}$ via $D_{s^*s} = \{a\}$, $D_{ss^*} = \{b\}$ and $D_0 = \varnothing$. Then

$X$ is domain-measurable as, for instance, $\mu = \delta_a$ defines a domain-measure, but $X$ cannot be amenable. Indeed, $D_0 = \varnothing = D_{s^*s} \cap D_{ss^*}$ and, hence, $\mu(D_0) = 1$ from the amenability of $X$, while $\mu(D_0) = 0$ since it is the empty set.

### 4.4.1 Semigroups coming from equivalence relations

Let $X$ be a discrete set, and let $\sim$ be an equivalence relation on $X$. There are several inverse semigroups one can construct from $\sim$, but here we only consider two of them.

The semigroup $S_\sim^{\mathrm{fin}}$ is the semigroup of *finite (i.e., open compact) bisections of* $\sim$, that is, the set of partial bijections $s\colon D_{s^*s} \to D_{ss^*}$, where $D_{s^*s}, D_{ss^*} \subset X$ are finite sets and $x \sim sx$ for every $x \in D_{s^*s}$. Note this inverse semigroup has a zero element $0 \in S_\sim^{\mathrm{fin}}$, namely the empty function, and operation given by composition whenever defined. Note, in addition, that the domain $D_0$ is the empty set. Lastly, observe that $S_\sim^{\mathrm{fin}}$ is represented in $\mathcal{I}(X)$ by construction.

From a semigroup-theoretical point of view the semigroup $S_\sim^{\mathrm{fin}}$ is rather trivial, for $s\mathcal{L}t$ whenever they have the same domains (and similarly for Green's right relation $\mathcal{R}$). Therefore the $\mathcal{H}$-class of an idempotent $e \in E(S_\sim^{\mathrm{fin}})$ is the symmetric group on $|D_e|$ elements. Moreover, observe that, from the point of view of this chapter, the semigroup $S_\sim^{\mathrm{fin}}$ is rather uninteresting.

**Proposition 4.4.3.** *Let $\sim$ be an equivalence relation on $X$, and let $S_\sim^{fin}$ be constructed as above. Then $S_\sim^{fin}$ is locally finite and, therefore, the canonical representation on $X$ is domain-measurable. However, this representation is not amenable, since* $0$ *defines the empty map on $X$.*

*Proof.* By Proposition 4.4.1 we only have to prove that $S_\sim^{\mathrm{fin}}$ is locally finite. Given any finitely generated sub-semigroup $T = \langle s_1, \ldots, s_k \rangle \subset S_\sim^{\mathrm{fin}}$ observe that $T$ naturally sits within the inverse symmetric monoid of $\cup_{i=1}^k D_{s_i^* s_i}$ elements, which is finite since the elements $s_i \in S_\sim^{\mathrm{fin}}$ have finite domains. □

In light of the latter proposition it is clear that the requirement that any $s \in S_\sim^{\mathrm{fin}}$ has finite domain is rather restricting. This raises the following alternative construction.

The semigroup $S_\sim$ of *(open) bisections of* $\sim$, that is, $S_\sim$ is the set of partial bijections $s\colon D_{s^*s} \to D_{ss^*}$, where $D_{s^*s}, D_{ss^*} \subset X$ and $x \sim sx$ for every $x \in D_{s^*s}$. Note the only difference between $S_\sim^{\mathrm{fin}}$ above and $S_\sim$ is that the domains of the elements in $S_\sim$ need not be finite and, therefore, the semigroup $S_\sim$ is, again, easily understood. However, the canonical representation of $S_\sim$ needs not be domain-measurable, as the following example shows.

**Example 4.4.4.** Let $X := \mathbb{N}$ and let $\sim$ be the trivial equivalence relation on $\mathbb{N}$ (every element is related to every other). We claim that, in this case, the canonical representation $S_\sim \to \mathcal{I}(\mathbb{N})$ is not domain-measurable. Indeed, partition $\mathbb{N} = E \sqcup O$ into the even and odd numbers, and let $s_1, s_2 \in S_\sim$ be the partial bijections with domain $\mathbb{N}$ and image $E$ and $O$ respectively. In particular $O = s_1 s_1^* O$ and $E = s_2 s_2^* E$ and, therefore

$$\mathbb{N} = O \sqcup E = s_1^* O = s_2^* E,$$

exhibiting a paradoxical decomposition of $\mathbb{N}$ under $\sim$.

The following definition encapsulates the idea of a *paradoxical decomposition* of $X$ using elements in $S_\sim$, and is inspired by the discussion of Ponzi schemes in the introduction.

**Definition 4.4.5.** Let $X$ be a discrete and countable set, and let $\sim$ be an equivalence relation on $X$. Given a subset $A \subset X$, we say $\sim$ *implements a Ponzi scheme on* $A$ if there are injective maps $\psi_1, \psi_2 \colon A \to A$ with disjoint ranges such that $x \sim \psi_1(x) \sim \psi_2(x)$ for all $x \in X$.

**Remark 4.4.6.** We have chosen the name *Ponzi scheme* because that is precisely what it is. Indeed, note that if $A$ is a set of people and $\sim$ is the relation *is a friend of ... of a friend of*, then the people of $A$ can agree on creating a Ponzi scheme, in the sense of creating money via moving it around, if and only if $\sim$ is as in Definition 4.4.5. In this case, moreover, if $\psi_1(x) = y$ then we would say that $y$ *gives* 1$ *to* $x$ (see also Figure 1.2). Therefore, and since for any $x \in A$ there are exactly two $y_1, y_2$ such that $\psi_1(y_1) = \psi_2(y_2) = x$, the person $x$ gives 1$ but receives 2$, yielding a net gain of 1$.

**Remark 4.4.7.** Observe, as well, the similarities between Definition 4.4.5 and the notion of properly infinite projection (see Definition 2.3.7).

**Theorem 4.4.8.** *Let $\sim$ be an equivalence relation on a discrete set $X$, and let $A \subset X$. Then the following assertions are equivalent:*

*(1) $A$ is $S_\sim$-domain measurable.*

*(2) $\sim$ does not implement a Ponzi scheme on $A$ (in the sense of Definition 4.4.5).*

*(3) Some $\sim$-class in $A$ is finite.*

*Proof.* (1) $\Rightarrow$ (2) follows from a contradiction argument, which leads to an inequality of the form $1 \geq 2$. In turn, if every $\sim$-class in $A$ is infinite then it is easy to construct a Ponzi scheme in $A$, proving that (2) $\Rightarrow$ (3). Finally, for (3) $\Rightarrow$ (1) let $A_0 \subset A$ be a finite $\sim$-class. Then the measure $\mu(B) := |B \cap A_0|/|A_0|$ witnesses the domain-measurability of $A$. □

### 4.4.2 Cuntz algebras and polycyclic monoids

Given $n \in \mathbb{N} \sqcup \{\infty\}$, recall that the *Cuntz algebra* $\mathcal{O}_n$ (see Cuntz's seminal work [27]) is the universal C*-algebra generated by $n$ isometries $s_1, \ldots, s_n$ such that $s_i^* s_j = 0$ unless $i = j$, and such that $1 = \sum_{i=1}^n s_i s_i^*$. This presentation of $\mathcal{O}_n$ is reminiscient of the so-called *polycyclic monoid* $\mathcal{P}_n$, that is, the semigroup given by the presentation:

$$\mathcal{P}_n := \{1, 0\} \sqcup \langle a_1, \ldots, a_n \mid a_i^* a_j = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{otherwise,} \end{cases} \rangle$$

where 1 and 0 represent the unit and the zero element of $\mathcal{P}_n$. The internal structure of the semigroup $\mathcal{P}_n$ is well understood. In order to go into details, let $A_n = \{a_1, \ldots, a_n\}$, and let $\mathcal{W}(A_n)$ be the set of words with letters in $A_n$, including the empty word $\lambda$. As a warning, note this notation is not commonly used within the semigroup community, but it is more adequate to our purposes.[2] First observe that any element $s \in \mathcal{P}_n$ can be reduced to a word of the form $\alpha\beta^*$, where $\alpha, \beta \in \mathcal{W}(A_n)$. In this representation the product in $\mathcal{P}_n$ follows the simple rule:

$$\alpha\beta^* \cdot \pi\rho^* = \begin{cases} \alpha\nu\rho^* & \text{if } \pi = \beta\nu, \\ \alpha\mu^*\rho^* & \text{if } \beta = \pi\mu, \\ 0 & \text{otherwise.} \end{cases}$$

Using these techniques, it is straightforward to represent $\mathcal{P}_n$ in $\mathcal{W}(A_n)$. Indeed, the following is known as the *Munn representation*

$$\mathcal{P}_n \to \mathcal{I}(\mathcal{W}(A_n)), \quad \text{where} \;\; \alpha\beta^* \colon \beta\mathcal{W}(A_n) \to \alpha\mathcal{W}(A_n), \quad \alpha\beta^*(\beta\omega) := \alpha\omega,$$

where $\beta\mathcal{W}(A_n) \subset \mathcal{W}(A_n)$ correspond to the words with letters in $A_n$ that start with $\beta$. Likewise, $0, 1 \in \mathcal{P}_n$ represent the identity and the empty map in $\mathcal{W}(A_n)$. The following proposition should be no surprise.

**Proposition 4.4.9.** *Let $n \in \mathbb{N} \sqcup \{\infty\}$. Then $\mathcal{W}(A_n)$ is domain measurable (under the Munn representation of $\mathcal{P}_n$) if, and only if, $n = 1$.*

*Proof.* If $n = 1$ note $\mathcal{P}_1$ is the bicyclic monoid with a zero element adjoined ad-hoc (see Example 3.3.3 and 5.1.6). It is then trivial to show that the finite sets

$$F_k := \left\{\lambda, a, \ldots, a^k\right\} \subset \mathcal{W}(A_1)$$

define a domain-Følner sequence in $\mathcal{W}(A_1)$ and the claim follows by Theorem 4.3.7.

Suppose now that $n \geq 2$. In this case

$$\mathcal{W}(A_n) = a_1^* a_1 \mathcal{W}(A_n) = a_2^* a_2 \mathcal{W}(A_n) \supset a_1\mathcal{W}(A_n) \sqcup a_2\mathcal{W}(A_n),$$

yielding a paradoxical decomposition of $\mathcal{W}(A_n)$. □

### 4.4.3 Semigroups coming from (partial) group actions

This last class of examples justifies the localization and domain-measurability conditions. Following the work of Exel (see [35] and references therein) we say that a discrete and countable group $G$ *acts partially* on a discrete set $X$ if for every $g \in G$ there is a partial bijection $\theta_g \colon D_{g^{-1}} \to D_g$, where $D_{g^{-1}}, D_g \subset X$, such that for all $g, h \in G$:

[2] In the semigroup community $\mathcal{W}(A_n)$ would be denoted by $A_n^*$, but the $^*$ notation is, for obvious, rather inadequate in our setting.

(i) $D_1 = X$ and $\theta_1 = id_X$, the identity map on $X$.
(ii) $\theta_h(D_{h^{-1}} \cap D_g) = D_h \cap D_{hg}$.
(iii) $\theta_h(\theta_g x) = \theta_{hg} x$ for all $x \in D_{g^{-1}} \cap D_{g^{-1}h^{-1}}$.

The point here is, of course, that not only can the domain $D_g$ be properly contained in $X$, but that the product $\theta_h \theta_g$ is only the restriction of $\theta_{hg}$ to a certain domain. Exel already introduced in [35, Definition 2.1] the inverse semigroup $\mathcal{S}(G)$ as the universal inverse semigroup with generators $\{[g]\}_{g \in G}$ and relations given by

(1) $[g^{-1}][g][h] = [g^{-1}][gh]$ and $[g][h][h^{-1}] = [gh][h^{-1}]$ for all $g, h \in G$,
(2) $[g][1] = [g] = [1][g]$ for all $g \in G$.

The next theorem was already proven by Exel in [35, Theorem 4.2].

**Theorem 4.4.10.** *Let $G$ be a group, and let $\mathcal{S}(G)$ be constructed as above. Given any set $X$, there is a one-to-one correspondence between the partial actions of $G$ on $X$ and the representations $\alpha: \mathcal{S}(G) \to \mathcal{I}(X)$.*

Following the latter theorem, the next proposition is straightforward to prove.

**Proposition 4.4.11.** *Given a partial action of $G$ on $X$, let $\alpha: \mathcal{S}(G) \to \mathcal{I}(X)$ be the associated representation. Then the following hold:*

*(1) There is a one-to-one correspondence between the $G$-invariant finitely additive probability measures of $X$ and the domain-measures of $X$ under $\mathcal{S}(G)$.*
*(2) There is a one-to-one correspondence between the probability measures of $X$ such that $\mu(g(A \cap D_{g^{-1}})) = \mu(A)$ for all $g \in G$ and $A \subset X$ and Day's invariant measures of $X$ under $\mathcal{S}(G)$.*

*Proof.* Note that a *$G$-invariant finitely additive probability measure of $X$* is a measure $\mu$ on $X$ such that if $A \subset D_{g^{-1}}$ then $\mu(A) = \mu(gA)$. This clearly yields a domain-measure of $X$. Moreover, note that, by Corollary 4.1.4, an invariant (in the sense of Day) measure $\mu$ satisfies that $\mu(g(A \cap D_{g^{-1}})) = \mu(A)$, as claimed. □

## 4.5 Groupoid amenability, *weak containment* and semigroup amenability

Inverse semigroups and groupoids, ever since the work of Paterson [81], go *hand in hand.* This brief last section of the chapter aims to relate the amenability of an inverse semigroup $S$ with the amenability of its associated *universal groupoid* $G_{\mathcal{U}}(S)$. First though, a warning and a spoiler. As a warning, everything in this section is already known, and we go through it in order to give a more complete picture. As a spoiler, the answer is a clear and categorical *no* (see Theorem 4.5.3).

Let us recall the construction of the universal groupoid $G_{\mathcal{U}}(S)$. For simplicity, we shall follow the notation in [37] (see also [81, 36, 3] and references therein). Fix an inverse semigroup $S$, and let $E$ be its set of idempotents. We say a *filter* on $E$ is a non-empty $\xi \subset E$ that is upwards closed, meaning $e \in \xi$ and $e \leq f \in E$ then $f \in \xi$; and closed under multiplication, that is, if $e, f \in \xi$ then $ef \in \xi$. Moreover, in case $S$ has a zero element 0 we require that $0 \notin \xi$. We shall denote the set of filters of $E$ by $\hat{E}_0$. Equipping $\hat{E}_0$ with pointwise convergence, or, equivalently, the relative topology of $\hat{E}_0$ as a subset of $\{0,1\}^E$, the space $\hat{E}_0$ is a locally compact, totally disconnected, second countable Hausdorff space. Moreover, $\hat{E}_0$ is compact whenever $S$ is unital, as then a filter $\xi$ is non-trivial if, and only if, $1 \in \xi$. Observe the semigroup $S$ acts on $\hat{E}_0$ in a natural way:

$$\theta_s \colon D^\theta_{s^*s} \to D^\theta_{ss^*}, \quad \theta_s(\xi) := \{e \in E \mid e \geq sfs^* \text{ for some } f \in \xi\},$$

where $D^\theta_e \subset \hat{E}_0$ is the set of all filters containing $e \in E$. The *universal groupoid of $S$* is the *groupoid of germs* of the action of $S$ on $\hat{E}_0$, that is

$$G_{\mathcal{U}}(S) := \left\{[s,\xi] \text{ where } s \in S \text{ and } \xi \in D^\theta_{s^*s}\right\},$$

where two germs $[s,\xi], [t,\zeta] \in G_{\mathcal{U}}(S)$ are equal if, and only if, $\xi = \zeta$ and $se = te$ for some $e \in E$ such that $e \in \xi$. Observe, for example, that the groupoid operations of $G_{\mathcal{U}}(S)$ are given by

$$[s,\xi]^{-1} = [s^*, \theta_s(\xi)]; \;\; r([s,\xi]) = \theta_s(\xi); \;\; d([s,\xi]) = \xi; \;\; [t, \theta_s(\xi)] \cdot [s,\xi] = [ts,\xi].$$

Finally, $G_{\mathcal{U}}(S)$ is equipped with the topology generated by sets of the form

$$\{[s,\xi] \mid \xi \in U\}, \text{ where } s \in S \text{ and } U \subset D^\theta_{s^*s} \text{ is open.}$$

The relevance of this construction comes from the fact that it englobes the representation theory of the semigroup itself. As a corollary, the reduced and full C*-algebras of $S$ can be seen as groupoid algebras. Indeed, see [81, Theorems 4.4.1 and 4.4.2] for a proof of the following.

**Theorem 4.5.1.** *Let $S$ be a countable inverse semigroup, and let $G_{\mathcal{U}}(S)$ be its universal groupoid. Then $C^*_r(S) = C^*_r(G_{\mathcal{U}}(S))$ and $C^*(S) = C^*(G_{\mathcal{U}}(S))$.*

Finally, recall (see, for instance, [1, Definitions 2.2.2 and 2.2.8] or [115, Definition 2.3]) that an étale groupoid $G$ is *amenable* if for any compact $K \subset G$ and $\varepsilon > 0$ there is a continuous compactly supported $\eta \colon G \to [0,1]$ such that

$$\left| 1 - \sum_{h \in G : s(h) = r(g)} \eta(h) \right| \leq \varepsilon \quad \text{and} \quad \sum_{h \in G : s(h) = r(g)} |\eta(h) - \eta(hg)| \leq \varepsilon$$

for all $g \in K$. As is well known, the notion of amenable groupoid generalizes the notion of amenable group in the following sense (see, e.g., [1, Corollary 6.2.14]).

**Theorem 4.5.2.** *Let $G$ be an étale groupoid. $G$ is amenable if, and only if, its reduced C*-algebra is nuclear.*

Coming back to inverse semigroups, there are several natural *amenability* notions:

(a) $S$ is amenable, in the sense of Day.
(b) The groupoid $G_{\mathcal{U}}(S)$ is amenable.
(c) $S$ satisfies the *weak containment property*, that is, the reduced and full semigroup C*-algebras of $S$ coincide (see [67]).

Observe that (a), (b) and (c) are equivalent when $S$ is a group. Therefore one may wonder whether the above conditions are related for general inverse semigroups. As it turns out, the answer is rather clear.

**Theorem 4.5.3.** *Let $S$ be a countable inverse semigroup. Then (b) implies (c), and no other implication between the conditions above holds in general.*

*Proof.* As we mentioned, the proof is well known, so we only cite the adequate results. The fact that (b) implies (c) is clear (follows from Theorem 4.5.2). Willett [115] provided a counterexample to the implication (c) $\Rightarrow$ (b) based on a previous construction of Higson, Lafforgue and Skandalis [48] (see Section 5.4.5 and Theorem 5.4.10). Of course, $S := \mathbb{F}_2 \sqcup \{0\}$ is amenable but has a non-nuclear reduced algebra, proving (a) does not imply (b) (this can also be done with a box space construction starting with a property (T) group). In addition, the reduced and full C*-algebras of $S$ do not coincide, giving a counterexample to (a) $\Rightarrow$ (c). Lastly, an example of Nica [72] (see also [3]) gives a non-amenable semigroup whose reduced algebra is the Cuntz-Toeplitz algebra, proving the claim. Indeed, the Cuntz-Toeplitz algebra is nuclear, and for this class of semigroups conditions (b) and (c) are equivalent. $\square$

Theorem 4.5.3 is not surprising. Roughly speaking, we observe that Day's amenability asserts the existence of an (amenable) trace of $C_r^*(S)$, while the amenability of the groupoid $G_{\mathcal{U}}(S)$ is related to its nuclearity. From this point of view it should not be a surprise that there is no general relation between these notions, since there are tracial non-nuclear algebras (such as $C_r^*(\mathbb{F}_2)$) and nuclear traceless algebras (such as $\mathcal{O}_2$).

# Chapter 5
# Inverse semigroups as metric spaces

The present chapter shifts the point of view with respect to Chapter 4. Up until now we have been considering a countable and discrete inverse semigroup $S$ and a representation $\alpha\colon S \to \mathcal{I}(X)$ on a discrete set $X$. However, and for reasons that will become apparent during the computations, in this chapter we only consider the canonical Wagner-Preston representation $S \to \mathcal{I}(S)$. In this sense, and for the rest of the thesis, we have that the domain of an element $s \in S$ is $D_{s^*s} = \{x \in S \mid s^*sx = x\} = s^*sS$, while its range is $D_{ss^*} = ss^*S$. Of course, left multiplication by $s$ still defines a bijection from $D_{s^*s}$ onto $D_{ss^*}$. Observe that, to the best of our abilities, we keep the convention that elements denoted by $s, t, \ldots$ *act on things*, while elements $x, y, \ldots$ are *acted on*.

Given a countable and discrete inverse semigroup $S$, in Chapter 4 we built and studied the C*-algebra $\mathcal{R}_S$ which, in a lot of senses, is the analogue for inverse semigroups of the uniform Roe algebra $\ell^\infty(G) \rtimes_r G$. Indeed, see Theorem 4.3.1 for a proof of the fact that

$$\mathcal{R}_S = C^*\left(\ell^\infty(S) \cdot \{V_s \mid s \in S\}\right) \cong \ell^\infty(S) \rtimes_r S.$$

Actually, the main point of Theorem 4.3.1 is that, even though the crossed product $\ell^\infty(S) \rtimes_r S$ sits naturally inside $\mathcal{B}(\ell^2 S \otimes \ell^2 S)$, one can faithfully represent the crossed product in $\mathcal{B}(\ell^2 S)$, yielding the algebra $\mathcal{R}_S$ (see Theorem 4.3.1). Observe, as well, that in the proof of Theorem 4.3.1 we already use the convention that the symbols $x, y$ represent both *points* and *elements* of $S$, meaning we can multiply them.

The starting point of this chapter is the realization that the C*-algebra $\mathcal{R}_S$ might actually *be* a uniform Roe algebra over a certain metric space. If $S$ happens to be a finitely generated group, for instance, then it is known that

$$\mathcal{R}_G \cong \ell^\infty(G) \rtimes_r G \cong C^*_u(\Lambda_G),$$

where $\Lambda_G$ is the Cayley graph of $G$ with respect to any finite and symmetric generating set (see Theorem 2.3.14). In this sense, one realizes that the, *a priori*, analytically defined crossed product $\ell^\infty(G) \rtimes_r G$ can be seen as a C*-algebra coming from an intrinsically algebraic (or geometric) object, namely the Cayley graph of $G$. This chapter is hence devoted to study when the algebra $\mathcal{R}_S \cong \ell^\infty(S) \rtimes_r S$ can be seen as the uniform Roe algebra of a metric space underlying the semigroup $S$.

First, in Section 5.1 we equip the inverse semigroup $S$ with a distance, generalizing the path distance in the Cayley graph of a group $G$. This distance, as we shall see, makes use of the *Schützenberger graphs* of the semigroup. Moreover, in sections 5.1.1 and 5.1.2 we prove some basic properties of this distance. Section 5.2 then studies the relation between the algebra $\mathcal{R}_S$ and the uniform Roe algebra $C_u^*(\Lambda_S)$, and characterizes exactly when these are equal since, as it turns out, they may not be the same algebra. Having equipped the inverse semigroup with a natural distance, Section 5.3 then studies some quasi-isometric invariants of the graph $\Lambda_S$ (or, equivalently, of the inverse semigroup $S$). Of particular interest are the *domain-measurability* of the semigroup (see Section 5.3.1) and the *property A* of the metric space $\Lambda_S$ (see Section 5.3.2). Section 5.4 then gives various interesting constructions that may be replicated in the context of inverse semigroups, providing numerous examples. Lastly, Section 5.5 relates the property A of the semigroup with the amenability of its universal groupoid.

## 5.1 Importance of extended metric spaces

Let us briefly state our current endeavor. Given a, say finitely generated, inverse semigroup $S$ we wish to construct an undirected graph $\Lambda_S$ such that the algebras $\mathcal{R}_S$ and $C_u^*(\Lambda_S)$ are equal (as subalgebras of $\mathcal{B}(\ell^2 S)$). Indeed, note that we aim for the path distance in $\Lambda_S$ to be symmetric, and hence we need $\Lambda_S$ to be undirected. Moreover, observe that the finite generation condition will ensure that $\Lambda_S$ is of bounded dimension, and our construction here will have to generalize the Cayley graph construction in the group case.

For the next, recall that we say a metric space $X$ is *extended* whenever two points $x, y \in X$ may be at infinite distance of each other (see Section 2.2). Likewise, any metric attaining the value $\infty$ is called an *extended* metric. The following proposition emphasizes the need to consider extended metric spaces in this context, and also serves as motivation behind Theorem 5.2.23.

**Proposition 5.1.1.** *Let $S$ be a countable inverse semigroup. Consider the matrix*

$$M_{x,y}: \ell^2(S) \to \ell^2(S), \quad \textit{where } x, y \in S \textit{ and } M_{x,y}(\delta_z) := \begin{cases} \delta_y & \textit{if } z = x \\ 0 & \textit{otherwise.} \end{cases}$$

*Let $d: S \to [0, \infty]$ be a (possibly extended) metric on $S$, and let $x, y \in S$.*

*(1) If $d(x, y) < \infty$ then $M_{x,y} \in C_u^*(S, d)$.*
*(2) If $x^* x \neq y^* y$ then $M_{x,y} \notin \mathcal{R}_S$.*

*Proof.* (1) follows from the fact that $M_{x,y} \in C_{u,alg}^*(X, d) \subset C_u^*(S, d)$. For (2), let $x, y \in S$ be such that $x^* x \neq y^* y$. Then the matrix unit $M_{x,y}$ is uniformly bounded away from any linear combination $\sum_{i=1}^{n} f_i V_{s_i} \in \mathcal{R}_{S,\mathrm{alg}}$:

$$\left\| M_{x,y} - \sum_{i=1}^{n} f_i V_{s_i} \right\|^2 \geq \left\| M_{x,y}(\delta_x) - \left( \sum_{i=1}^{n} f_i V_{s_i} \right)(\delta_x) \right\|^2$$
$$= \left\| \delta_y - \sum_{\substack{i=1 \\ x \in D_{s_i^* s_i}}}^{n} f(s_i x)\, \delta_{s_i x} \right\|^2 = 1 + \left\| \sum_{\substack{i=1 \\ x \in D_{s_i^* s_i}}}^{n} f(s_i x)\, \delta_{s_i x} \right\|^2 \geq 1 \,,$$

where the second equality follows from the fact that there can be no $s_i$ such that $x \in D_{s_i^* s_i}$ and $s_i x = y$, because otherwise $y^* y = x^* s_i^* s_i x = x^* x$, contradicting the hypothesis. □

The preceding proposition suggests that any pair $x, y \in S$ such that $x^* x \neq y^* y$ must be at infinite distance if one wishes to have $\mathcal{R}_S \cong C_u^*(\Lambda_S)$. Following this idea, we recall (see the discussion after Proposition 3.3.5) that $x, y \in S$ are $\mathcal{L}$*-related* if $x^* x = y^* y$, and that they are $\mathcal{R}$*-related* if $x x^* = y y^*$. Likewise, Green's relation $\mathcal{H}$ is defined as the intersection of $\mathcal{L}$ and $\mathcal{R}$, while $\mathcal{D}$ is their join. In particular, it is well known that two idempotents $e, f \in E(S)$ are $\mathcal{D}$-related precisely when $e = s^* s$ and $f = s s^*$ for some $s \in S$.

### 5.1.1 Schützenberger graphs

Fix a countable and discrete inverse semigroup $S$, and fix a symmetric generating set $K = K^* \subset S$. Note that, in general, we do not assume $K$ to be finite, nor do we assume $S$ to have a unit. Given an $\mathcal{L}$-class $L \subset S$, recall that the *left Schützenberger graph* of $L$ is the edge-labeled undirected graph whose vertex set is $L$, and such that two vertices $x, y \in L$ are joined by an edge labeled by $k \in K$ if $kx = y$ (see, e.g., [98, 99, 41]). Observe that for inverse semigroups $\Lambda(L, K)$ is an undirected graph, in the sense that if $x, y \in L$ and $kx = y$ then $y^* y = x^* k^* k x$ and, thus, $k^* y = k^* k x = x$. Therefore there is a $k^*$-labeled edge going from $y$ to $x$. We will denote by $\Lambda(S, K)$ the disjoint union of the left Schützenberger graphs $\Lambda(L_e, K)$, where $e \in E(S)$. That is, the vertex set of $\Lambda(S, K)$ is $S$ and two vertices $x, y \in S$ are joined by an edge labeled by $k \in K$ if and only if $x \mathcal{L} y$ and $kx = y$.

**Remark 5.1.2.** Note that $\Lambda(S, K)$ is not the usual (left) Cayley graph of a semigroup (see, e.g., [41]). Indeed, observe that the Cayley graph of $S$ is in general a directed graph while $\Lambda(S, K)$ is always undirected. For instance, if $S = \{0, 1\}$ equipped with the product, then its Cayley graph has a directed edge going from 1 to 0, while in $\Lambda(S, K)$ the vertices 0 and 1 are in different connected components. In fact, it can be shown that $\Lambda(S, K)$ is the graph resulting from deleting the directed edges in the left Cayley graph of $S$.

Alternatively one could construct the undirected graph $\Sigma(S, K)$, whose connected components are the *right Schützenberger graphs* $\Sigma(R, K)$ of each $\mathcal{R}$-class $R$.

For general semigroups the left and right version of these graphs need not even be coarsely equivalent, since an arbitrary semigroup could, for instance, have a distinct number of $\mathcal{L}$ and $\mathcal{R}$ classes (cf., [41, Example 1]). However, for inverse semigroups these graphs are isomorphic.

**Lemma 5.1.3.** *The graphs $\Lambda(S,K)$ and $\Sigma(S,K)$ are isomorphic (as graphs).*

*Proof.* The isomorphism of graphs is given by the involution map, which is clearly bijective and maps $\mathcal{L}$-classes onto $\mathcal{R}$-classes. In addition, since the generator set is symmetric, we have that adjacency relations of the graphs are also preserved, since $y = kx$ if and only if $y^* = x^* k^*$. □

For the sake of simplicity, we will henceforth omit the term *left* when referring to Schützenberger graphs. Furthermore, if the generating set $K$ is clear from the context we will denote $\Lambda(S,K)$ just as $\Lambda_S$. Similarly, for any $\mathcal{L}$-class $L \subset S$ we will often write its Schützenberger graph simply by $\Lambda_L$. We will also refer in some situations (specially in Section 5.3) to the inverse semigroups as metric spaces, by which we mean the corresponding disjoint union of Schützenberger graphs equipped with the usual path length metric.

**Remark 5.1.4.** A point $x \in S$ may be connected with itself with a non-trivial loop, that is, an edge starting and ending at $x$ and labeled by an element $k \in K$ that is not the unit. For instance, let $S = G \sqcup H$, where $h \cdot g = g \cdot h := g$ for any $g \in G$ and $h \in H$. Then the only $\mathcal{L}$-classes of $S$ are $G$ and $H$, and any $g \in G$ is decorated with a loop in $\Lambda_S$ labeled by each element $h \in H$.

**Remark 5.1.5.** Infinitely many edges might connect two vertices $x, y \in S$. Indeed, let $S := G \times \mathbb{N}$, where $G$ is a discrete group generated by $K$ and the operation is given by

$$(g,n) \cdot (h,m) := (gh, \min\{n,m\}) .$$

Then $S$ is an inverse semigroup with $(g,n)^* := (g^{-1}, n)$, and any $(g,1), (kg,1) \in G \times \{1\}$ are connected by infinitely many edges of the form $(k,m) \in K \times \mathbb{N}$.

The following is the main example of importance (see also Section 5.4).

**Example 5.1.6.** Let $\mathcal{T} = \langle a, a^* \mid a^* a = 1 \rangle$ be the bicyclic monoid (see Example 3.3.3). Note that the elements of $\mathcal{T}$ are words of the form $a^i a^{*j}$, where $i, j \geq 0$. Of these, the idempotents are exactly those for which $i = j$. Moreover, observe that $a^i a^{*j} \mathcal{L} a^p a^{*q}$ if, and only if, $j = q$. Therefore, if one chooses the natural generating set $K := \{a, a^*\}$, the resulting graph $\Lambda_{\mathcal{T}}$ consists of countably many half-lines (see Figure 5.1).

### 5.1.2 *Falling down* of the paths

This section is, in essence, the crucial Lemma 5.1.7. Recall that the *maximal homomorphic image* of $S$ is the quotient $S/\sigma$, where $s \sigma t$ if, and only if, $se = te$ for some

$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \iddots \\
& & & & \\
a^3 & a^3a^* & a^3a^{*2} & a^3a^{*3} & \cdots \\
| & | & | & | & \\
a^2 & a^2a^* & a^2a^{*2} & a^2a^{*3} & \cdots \\
| & | & | & | & \\
a & aa^* & aa^{*2} & aa^{*3} & \cdots \\
| & | & | & | & \\
1 & a^* & a^{*2} & a^{*3} & \cdots
\end{array}
$$

FIGURE 5.1: Graph $\Lambda_{\mathcal{T}}$ associated to the biclycic monoid $\mathcal{T} = \langle a, a^* \mid a^*a = 1\rangle$ with respect to the canonical generating set $K := \{a, a^*\}$. Any two half-lines are at infinite distance from one another.

idempotent $e \in E(S)$. As can easily be proven, the quotient, denoted by $G(S)$, is always a group. Moreover, observe that if $S$ is generated by $K$ then $G(S)$ is generated by $\sigma(K)$, where $\sigma : S \to G(S)$ denotes the canonical projection (note the slight abuse of notation). In this sense, one can equip $G(S)$ with the path metric $d_{G(S)}$ coming from the Cayley graph of $G(S)$ with respect to $\sigma(K)$. Likewise, let $d_S$ stand for the path metric in $\Lambda_S$. Note, in particular, that $d_S(x, y) = \infty$ whenever $x^*x \neq y^*y$. Lastly, $\ell(s)$ denotes the length of an element $s \in S$ in the alphabet $K$.

**Lemma 5.1.7.** *Let $S = \langle K\rangle$ be an inverse semigroup, and let $\sigma : S \to G(S)$ be the canonical projection onto the maximal homomorphic image.*

*(1) For any $s, x \in S$ such that $x \in D_{s^*s}$ we have*

$$d_{G(S)}\left(\sigma\left(x\right), \sigma\left(sx\right)\right) \leq d_S\left(x, sx\right) \leq d_S\left(s^*s, s\right) \leq \ell(s).$$

*(2) For any $s, x \in S$ such that $xx^* = s^*s$ (hence, in particular, $x \in D_{s^*s}$), we have*

$$d_S\left(x, sx\right) = d_S\left(s^*s, s\right).$$

*Proof.* For (1) observe that if $\ell(s) = d$ and $s = k_d \ldots k_1 \in K^d$, then $s^*s$ and $s$ are joined by a path labeled by $k_d, \ldots, k_1$, and thus $\ell(s) \geq d_S(s^*s, s)$. Moreover, for $x \in D_{s^*s}$ a geodesic path joining $s^*s$ with $s$ will define by multiplication from the right with $x$ a path of the same length joining $x$ with $sx$ on $L_{x^*x}$ and, therefore, $d_S(x, sx) \leq d_S(s^*s, s)$. Similarly, any geodesic path joining $x$ with $sx$ will define via the quotient map $\sigma$ a path joining $\sigma(x)$ with $\sigma(sx)$ in the Cayley graph of $G(S)$, proving the last inequality.

Part (2) follows from (1) since

$$d_S\left(x, sx\right) \leq d_S\left(s^*s, s\right) = d_S\left(xx^*, sxx^*\right) \leq d_S\left(x, sx\right),$$

where the last inequality is, again, due to the fact that a geodesic connecting $x$ and $sx$ in $\Lambda_S$ defines a path between $xx^*$ and $sxx^*$ when multiplied on the right by $x^*$. □

**Remark 5.1.8.** Observe that Lemma 5.1.7 defines precisely the meaning of *right invariance* in the context of inverse semigroups. Recall that a metric $d$ in a group $G$ is said to be *right invariant* if $d(g,h) = d(gx,hx)$ for every $g,h,x \in G$. Moreover, note that such an exact invariance condition cannot be attained for inverse semigroups. Indeed, let $S = \mathbb{Z} \times \{1,2\}$, where $(n,i) + (m,j) := (n+m, \min\{i,j\})$. Equip $S$ with the generating set $K := \{(\pm 2,1), (\pm 3,1), (\pm 1,2)\}$. Then

$$d\big(\,(0,2)+(0,1)\,,(2,2)+(0,1)\,\big) = d\big(\,(0,1)\,,(2,1)\,\big) = 1 < 2 = d\big(\,(0,2)\,,(2,2)\,\big).$$

In general, observe that, given $s,t,x \in S$

$$(tx)\,(sx)^* = txx^*s^* = txx^*s^*ss^* = (ts^*)\cdot(sxx^*s^*)$$

and hence $(tx)(sx)^* \leq ts^*$, which we see as the pair $(sx,tx)$ being *below* the pair $(s,t)$. Therefore, and roughly speaking, any path connecting $s$ and $t$ *falls down* to a path connecting $sx$ and $tx$. Furthermore, we use the verb *falls down* because paths *below* may not lift to paths *above*. Therefore, in a sense, the points of $S$ are *more connected* in the $\mathcal{L}$-classes that correspond to small idempotents than they are in the $\mathcal{L}$-classes that correspond to large idempotents. However, note that, due to the non-injectivity of left multiplication, the $\mathcal{L}$-classes that correspond to small idempotents may be smaller (in size) than the ones corresponding to large idempotents, meaning that different paths above label the same path on a level below.

The following proposition proves that some of the Schützenberger graphs $\Lambda_L$ are isomorphic (as graphs) to each other.

**Proposition 5.1.9.** *Let $S$ be a countable inverse semigroup. Given $s \in S$, let $\Lambda_{s^*s}$ and $\Lambda_{ss^*}$ be the Schützenberger graphs of the $\mathcal{L}$-classes of $s^*s$ and $ss^*$, respectively. Then*

$$\rho\colon \Lambda_{s^*s} \to \Lambda_{ss^*}, \quad \text{where } \rho(x) := xs^*$$

*defines a graph isomorphism between $\Lambda_{s^*s}$ and $\Lambda_{ss^*}$.*

*Proof.* First note that $(xs^*)^*xs^* = sx^*xs^* = ss^*ss^* = ss^*$ and thus $xs^*\mathcal{L}ss^*$. If $xs^* = ys^*$ then $x = xx^*x = xs^*s = ys^*s = yy^*y = y$ for any $x,y \in \Lambda_{s^*s}$, which proves that $\rho$ is injective. Given an arbitrary $t \in \Lambda_{ss^*}$, observe that $t^*t = ss^*$ and, therefore, $t = tss^* = \rho(ts)$ since $ts\mathcal{L}s^*s$, proving that $\rho$ is a bijection. Finally, since $\rho$ is defined by right multiplication it is clear that it preserves adjacency. Indeed, the points $x, kx \in \Lambda_{s^*s}$ are joined by an edge labeled by $k$ if and only if the points $xs^*, kxs^* \in \Lambda_{ss^*}$ are joined by an edge labeled by $k$. □

From the language of semigroup theory, Proposition 5.1.9 precisely states that if $e,f \in E(S)$ are $\mathcal{D}$-related then their Schützenberger graphs $\Lambda_e$ and $\Lambda_f$ are isomorphic as graphs. For instance, in the particular case of the bicyclic monoid $\mathcal{T}$ (see

Example 5.1.6), the latter proposition gives that right multiplication by $a^*$ defines a graph isomorphism between the (left) Schützenberger graph of 1 and the (left) Schützenberger graph of $a^*$ (this isomorphism is, of course, the obvious one).

## 5.2 Large scale geometry of an inverse semigroup

From now on, let $S$ be a discrete and countable inverse semigroup, and fix a symmetric and generating set $K = K^* \subset S$. This section aims to study some basic large-scale properties of the graph $\Lambda_S$.

**Proposition 5.2.1.** *Let $S = \langle K \rangle$ be a finitely generated inverse semigroup, with $K$ finite. Then $\Lambda_S$ is of bounded geometry.*

*Proof.* For any $x \in S$ and $r > 0$ we have the uniform estimate $|B_r(x)| \leq |K|^r$. □

Observe that our setting is that of semigroups which are countable but not necessarily finitely generated. We do, however, require $\Lambda_S$ to be of bounded geometry, in which case our setting reduces to the usual one in groups. Recall that the Cayley graph of a group $G$ is bounded geometry if and only if $G$ is finitely generated.

**Proposition 5.2.2.** *Let $S = \langle K \rangle$ be an inverse semigroup. If $\Lambda_S$ is of bounded geometry, then so is the left Cayley graph of the maximal homomorphic image $G(S)$.*

*Proof.* Given $r > 0$, since $\Lambda_S$ has bounded geometry, we have that

$$m \coloneqq \sup_{x \in S} |B_r(x)| < \infty \,.$$

We claim that $m$ also uniformly bounds the cardinality of the $r$-balls in the Cayley graph of $G(S)$, thus proving that $G(S)$ has bounded geometry. Indeed, assume that this is not the case. Let $\sigma \colon S \to G(S)$ be the canonical quotient map and let $B_r(\sigma(x_1))$ be an $r$-neighborhood (with respect to the path distance) of some $\sigma(x_1) \in G(S)$ having at least $m+1$ different points, i.e.,

$$\{\sigma(x_1), \ldots, \sigma(x_{m+1})\} \subset B_r(\sigma(x_1)) \,.$$

Any point $\sigma(x_i)$ is connected with $\sigma(x_1)$ by a geodesic path $\sigma(s_i) \in G(S)$ of length at most $r$, and in particular $\sigma(x_i) = \sigma(s_i x_1)$ for every $i = 2, \ldots, m+1$. Consider then the idempotent given by

$$e \coloneqq x_1^* s_2^* s_2 x_1 \, \ldots \, x_1^* s_{m+1}^* s_{m+1} x_1 ,$$

and let $y_1 \coloneqq x_1 e$ and $y_i \coloneqq s_i y_1$, where $i = 2, \ldots, m+1$. We claim that the points $\{y_i\}_{i=1}^{m+1}$ are pairwise different and within distance $r$ of $y_1$, thus proving that $m$ is not a bound on the cardinality of the $r$-balls of $S$ and contradicting our assumption.

Indeed, first note that, when $i \neq j$, it follows that $y_i \neq y_j$ since $\sigma(y_i) = \sigma(s_i x_1) = \sigma(x_i) \neq \sigma(x_j) = \sigma(s_j x_1) = \sigma(y_j)$. Secondly, observe

$$s_i^* s_i y_1 = s_i^* s_i x_1 e = s_i^* s_i x_1 x_1^* s_i^* s_i x_1 e = x_1 x_1^* s_i^* s_i x_1 e = x_1 e = y_1$$

and hence $y_1 \in D_{s_i^* s_i}$ for every $i = 2, \ldots, m+1$, proving that $y_1, \ldots, y_{m+1}$ are all $\mathcal{L}$-related (cf., Lemma 4.3.3). Finally, the points $y_1$ and $y_i$ are connected by a path of length less than $r$ by construction, since $s_i y_1 = y_i$, proving the claim. □

**Remark 5.2.3.** The difficulty of the preceding proof is the fact that the elements $x_1, \ldots, x_{m+1} \in S$ need not be $\mathcal{L}$-related, that is, they might sit in different Schützenberger graphs of $S$. Moreover, the edges connecting $\sigma(x_1), \ldots, \sigma(x_{m+1})$ in $G(S)$ need not be present in a certain $\mathcal{L}$-class, and thus we have to move the point $x_1 \in S$ down via multiplication with a suitable projection $e$ in order to replicate those edges in a certain $\mathcal{L}$-class $L_e \subset S$. Therefore, what the proof above actually says is that the local structure of $G(S)$, i.e., a certain $r$-ball $B_r(\sigma(x_1)) \subset G(S)$, may be seen in an $\mathcal{L}$-class $L_e \subset S$, provided that $e \in E(S)$ is a sufficiently small idempotent. In particular, the left Cayley graph of $G(S)$ is the inductive limit of the Schützenberger graphs of $S$ (see also the proof of Proposition 5.3.11).

The following proposition generalizes a well known result in the case of groups (see [76, Theorem 1.3.12]). See also [41, Proposition 4] for a similar statement for semigroups considered as semi-metric spaces.

**Proposition 5.2.4.** *Let $S$ be an inverse semigroup and let $K_1, K_2 \subset S$ be two finite and symmetric generating sets. The graphs $\Lambda(S, K_1)$ and $\Lambda(S, K_2)$ are quasi-isometric.*

*Proof.* The identity function $id\colon \Lambda(S, K_1) \to \Lambda(S, K_2)$ is a quasi-isometry. Indeed, since it is surjective it is enough to prove there is some constant $m \geq 1$ such that

$$\frac{1}{m} d_{K_1}(x, y) \leq d_{K_2}(x, y) \leq m d_{K_1}(x, y) \quad \text{for any } x, y \in S,$$

where $d_{K_i}$ denotes the path distance in the graph $\Lambda(S, K_i)$. Note that if $x$ and $y$ are not $\mathcal{L}$-related then $d_{K_1}(x, y) = d_{K_2}(x, y) = \infty$, so we may suppose that $x, y$ belong to the same $\mathcal{L}$-class. In this case the inequalities follow by letting

$$m := \max\{d_{K_1}(k^* k, k)\,,\, d_{K_2}(k^* k, k) \mid k \in K_1 \cup K_2\}.$$

□

### 5.2.1 E-unitary and F-inverse cases: upper and lower bounds

This section follows the study of the large-scale geometry of $S$ (or, rather, the graph $\Lambda_S$) in relation with that of the maximal homomorphic image $G(S)$.

The first observation is that $G(S)$ might be radically different from $S$ from a geometrical point of view. Indeed, any $S = H \sqcup \{0\}$, where $H$ is an infinite group, has trivial homomorphic image, but $\Lambda_S$ isometrically contains the Cayley graph of $H$. This behavior happens because of the germ-like nature of $\sigma$ and, in particular, because left multiplication by idempotents need not define an injective map in $S$. The following class of semigroups avoids this kind of behavior.

**Definition 5.2.5.** An inverse semigroup $S$ is *E-unitary* if the canonical projection $\sigma: S \to G(S)$ is idempotent pure, that is, $E(S) = \sigma^{-1}(1)$.

**Remark 5.2.6.** Observe that $\sigma(e) = \sigma(f)$ for all $e, f \in E(S)$ and, thus, $E(S) \subset \sigma^{-1}(1)$. The semigroup $S$ is then E-unitary whenever $s\sigma e \in E(S)$ implies $s \in E(S)$.

The following proposition is one of the many characterizations of E-unitary semigroups, see Lawson's book [58, Section 2.4, Theorem 6] for its proof and for some other approaches.

**Proposition 5.2.7.** *A semigroup $S$ is E-unitary if, and only if, the projection $\sigma: S \to G(S)$ restricts to an injective map in every $\mathcal{L}$-class of $S$.*

Using the latter and Lemma 5.1.7 yields the following.

**Corollary 5.2.8.** *Let $S$ be an E-unitary inverse semigroup, and let $L \subset S$ be an $\mathcal{L}$-class. Then $\sigma: S \to G(S)$ is injective in $L$ and $d_{G(S)}(\sigma(x), \sigma(y)) \leq d_S(x, y)$ for every $x, y \in L$.*

Observe that, if $S$ is E-unitary, then we know that $\sigma$ is injective in every $\mathcal{L}$-class, but it need not be a quasi-isometric embedding. Indeed, see Section 5.4.4 for an example of an E-unitary inverse semigroup such that $\sigma$ is not even a coarse embedding.

The E-unitary notion, and the idea that the distance $d(x, y)$ can be bounded below, is further strengthened by the following proposition. First recall that, if $(\mathcal{P}, \leq)$ is a partially ordered set, we say an element $p \in \mathcal{P}$ is *maximal* if there is no element $q \in \mathcal{P}$ such that $q \geq p$ and $q \neq p$. Likewise, we say $p$ is *greatest* if $p \geq q$ for all $q \in \mathcal{P}$. Observe that every greatest element is automatically maximal but a maximal element need not be greatest. Observe, moreover, that a partially ordered set need not have any maximal element and neither does it need to have a greatest element, even if there are maximal elements. Lastly, recall that we say $s \leq t$, for $s, t \in S$, whenever $ts^*s = s$, and note that this partial order is a partial order within the $\sigma$-classes of $S$.

**Definition 5.2.9.** An inverse semigroup $S$ is *F-inverse* if the $\sigma$-class of any element $s \in S$ has a unique greatest element.

F-inverse semigroups have been studied in the literature, see, for instance, Lawson's [58, Section 7.4] for a comprehensive introduction. In particular, the following is well known (cf., [58, Proposition 3, Chapter 7]).

**Proposition 5.2.10.** *Any F-inverse semigroup is E-unitary. Moreover, the bicyclic monoid (see Example 5.1.6) is F-inverse.*

From a metrical point of view these F-inverse semigroups are interesting because their distance can be bounded above.

**Corollary 5.2.11.** *Let $S$ be an F-inverse semigroup. Given any $s \in S$ denote by $m_s \in S$ the greatest element in the $\sigma$-class of $s$. Then:*

$$d_{G(S)}\left(1_{G(S)}, \sigma(s)\right) \leq d_S(s^* s, s) \leq d_S(m_s^* m_s, m_s).$$

*Proof.* The first inequality directly follows from Lemma 5.1.7, while the second follows by the same lemma by letting $x := s$ and noting that $s \leq m_s$. □

Another, perhaps more interesting, corollary is that the canonical projection map $\sigma$ coarsely embeds any Schützenberger graph of $S$ into $G(S)$ whenever $S$ is F-inverse. Indeed, note that the proof of the following makes explicit use of both the bounds *above* and *below* given in Corollary 5.2.11, along with the running assumption that the Schützenberger graphs of $S$ are of bounded geometry.

**Proposition 5.2.12.** *Let $S$ be an F-inverse semigroup, and let $L \subset S$ be an $\mathcal{L}$-class of bounded geometry equipped with the path metric of its associated Schützenberger graph. Then the quotient map $\sigma: S \to G(S)$ coarsely embeds $L$ into $G(S)$.*

*Proof.* Given a radius $r > 0$, let

$$\rho_+(r) := \sup\left\{d_S(s^* s, s) \mid \sigma(s) \in B_r(1) \subset G(S)\right\} \in [0, \infty],$$

where $B_r(1)$ denotes the $r$-ball around 1 in $G(S)$. It is then clear from Corollary 5.2.11 that

$$d_{G(S)}(1, \sigma(s)) \leq d_S(s^* s, s) \leq \rho_+\left(d_{G(S)}(1, \sigma(s))\right). \tag{5.1}$$

Moreover, Eq. (5.1) above may be improved to all pairs $x, y \in S$ noting that $d_S(x,y) = d_S(xx^*, yx^*)$ (see Lemma 5.1.7), and hence $\sigma$ is a coarse embedding provided that $\rho_+(r)$ is finite. To that end note $G(S)$ is of bounded geometry (see Proposition 5.2.2) and, therefore, the ball

$$B_r(1) = \{\sigma(s_1), \ldots, \sigma(s_k)\}$$

is a finite set. Let $m_1, \ldots, m_s \in S$ be the unique maximal elements above $s_1, \ldots, s_k$, that exist since $S$ is F-inverse. Then, following Lemma 5.1.7,

$$\rho_+(r) = \max_{i=1,\ldots,k}\{d_S(m_i^* m_i, m_i)\} < \infty,$$

which concludes the proof. □

### 5.2.2 The *finite labeling* condition, or when $\mathcal{R}_S \cong C^*_u(S)$

In this section we will show that the graph $\Lambda_S := \sqcup_e L_e$ (seen as a metric space with the path length metric) allows to witness the algebra $\mathcal{R}_S$ as the uniform Roe algebra $\mathrm{C}^*_u(\Lambda_S)$. We begin introducing an important notion for the semigroup $S$ (see in relation with Theorem 5.2.23).

**Definition 5.2.13.** Let $S = \langle K \rangle$ be an inverse semigroup with countable and symmetric generating set $K$ and let $L \subset S$ be an $\mathcal{L}$-class.

(1) We say the pair $(L, K)$ admits a *finite labeling*, or has condition FL for short, if there are $c \geq 1$ and a finite $K_1 \subset K$ such that for any $x, y \in L$ with $y \in Kx$, we have that $y \in K_1^p x$ for some $p \in \{1, \dots, c\}$, where $K_1^p$ denotes the words of length $p$ in the alphabet $K_1$.

(2) We say the pair $(S, K)$ admits a *finite labeling*, or has condition FL for short, if every $\mathcal{L}$-class of $S$ admits a finite labeling uniformly among the $\mathcal{L}$-classes, that is, if there are $c \geq 1$ and a finite $K_1 \subset K$ such that, for any $x, y \in S$ with $x\mathcal{L}y$ and $y \in Kx$, we have that $y \in K_1^p x$ for some $p \in \{1, \dots, c\}$.

**Remark 5.2.14.** We say *the pair $(S, K)$ has condition FL* instead of $S$ *has condition FL* since admitting a finite labeling may depend on the generating set $K$. However, as most times $K$ is fixed, we shall normally say $S$ *admits a finite labeling*, meaning the pair $(S, K)$ admits a finite labeling. Note, moreover, that inverse semigroups may have more than one proper and right invariant metrics, even up to coarse equivalence, contrary to the group case.

The preceding definition, although technical, is essential to various arguments in the chapter. It is, in particular, an algebraic characterization of the equality of C*-algebras $\mathcal{R}_S = \mathrm{C}^*_u(\Lambda_S)$ (see Theorem 5.2.23).

The following gives sufficient conditions for an inverse semigroup to admit a finite labeling and, in particular, shows there are important classes of examples which satisfy this condition.

**Proposition 5.2.15.** *The following classes of inverse semigroups are FL:*

*(1) The class of finitely generated inverse semigroups.*

*(2) The class of F-inverse semigroups such that $\Lambda_S$ is of bounded geometry.*

*Proof.* It is clear that every finitely generated inverse semigroup admits a finite labeling: take $c := 1$ and $K_1 := K$, which is finite by assumption. For the second statement let $S = \langle K \rangle$ be an F-inverse semigroup such that $\Lambda_S$ is of bounded geometry. Denote the maximal group homomorphic image $G(S)$ simply by $G$. Consider then the projection $\sigma$ restricted to the following set of greatest elements:

$$A_{\mathrm{great}} := \{ s \in S \mid s \text{ is greatest in } \sigma(s) \text{ and } \exists x \in D_{s^*s} \text{ with } d_S(x, sx) \leq 1 \},$$

i.e., consider

$$\sigma\colon A_{\mathrm{great}} \to \{\, g \in G \mid d_G(1_G, g) \le 1 \,\} = B_1(1_G). \tag{5.2}$$

Note that $A_{\mathrm{great}}$ is not empty because $1 \in A_{\mathrm{great}}$ and since the greatest element in each class is unique the preceding map is injective. Moreover, the right hand side is a finite set since $\Lambda_S$ is of bounded geometry (see Proposition 5.2.2) and, hence, $A_{\mathrm{great}}$ is finite too. Therefore there is a finite $K_1 \subset K$ such that every element in $A_{\mathrm{great}}$ is a word in $K_1$ of length less than $c := \max\{\ell(s) \mid s \in A_{\mathrm{great}}\}$. It can then be checked that these form a finite labeling of $S$. In fact, let $x\mathcal{L}y$ with $y = kx$ for some $k \in K$ (i.e., $d_S(x, y) \le 1$). Denote by $s_0$ the greatest element in $\sigma(k)$. Since $D_{k^*k} \subset D_{s_0^* s_0}$ we have $s_0 x = kx = y$ and, therefore, $y \in K_1^p x$ for some $p \in \{1, \ldots, C\}$, proving that $S$ is FL. □

**Remark 5.2.16.** Following the same argument as in Proposition 5.2.15 one can arrive at a broader class of FL semigroups. Indeed, observe that the key point in the proof is that the map $\sigma$ in Eq. (5.2) has finite image and finite preimages and, therefore, its domain is finite as well. Hence any semigroup with these same properties will be FL by the same argument.

In particular, we may say that $S$ is *finitely bounded above* if every $\sigma$-class of $S$ has a finite number of maximal elements that bound every element in the class. By the same argument, one can see these finitely bounded above semigroups also admit finite labelings. However, note the two notions, namely *finite bounded above* and *admitting a finite labeling*, are not equivalent. For example, any $S = G \sqcup \{0\}$, with $G$ a finitely generated infinite group, admits a finite labeling but is not finitely bounded above. We will not use the *finitely bounded above* notion in this thesis.

The following proposition gives necessary conditions for an inverse semigroup to be admit a finite labeling. Even though its proof is straightforward, it highlights some key ideas behind the definition. Note that the construction of $\Lambda(S, K)$ can be done similarly for any subset of labels $K_1 \subset K$, and the corresponding graph will be denoted by $\Lambda(S, K_1)$, which is then a subgraph of $\Lambda(S, K)$.

**Proposition 5.2.17.** *Let $S = \langle K \rangle$ be an inverse semigroup admitting a finite labeling, and let $K_1 \subset K$ be as in Definition 5.2.13. Then the following hold:*

*(1) The identity map on vertices $\Lambda(S, K) \to \Lambda(S, K_1)$ is a quasi-isometry.*

*(2) Every $s \in S$ can be written as a word $s = k_d \ldots k_1 e$, where $e \in E(S)$ and $k_i \in K_1$, that is, $S$ is finitely generated modulo idempotents.*

*Proof.* (1) follows from the fact that for every $x, y \in S$

$$d_K(x, y) \le d_{K_1}(x, y) \le l d_K(x, y)$$

where $d_K$ and $d_{K_1}$ denote the path distances in $\Lambda(S, K)$ and $\Lambda(S, K_1)$, respectively, and $l := c \ge 1$ is as in Definition 5.2.13. (2) holds since, by the hypothesis, any $s \in S$ can be written as $s = k_d \ldots k_1$ where $k_i \in K_1$, that is, $s = ss^*s = k_d \ldots k_1 s^* s$. □

Lastly, the following proposition is straightforward to prove, but we include it for later use.

**Proposition 5.2.18.** *Let $S = \langle K \rangle$ be an inverse semigroup. The pair $(S, K)$ admits a finite labeling if, and only if, for every $r \geq 0$ there is a finite $F_r \subset S$ such that if $d(s^*s, s) \leq r$ then there is some $m \in F_r$ such that $ms^*s = s$ (or, equivalently, $m \geq s$).*

*Proof.* Let $K_1 \subset K$ and $c \geq 1$ witness the FL condition of $(S, K)$ as in Definition 5.2.13. Given a radius $r > 0$ the set $F_r := K_1^{rc}$ is finite. Moreover, it labels all paths in $S$ in alphabet $K$ and of length at most $r$. That is, whenever $d(s^*s, s) \leq r$ there is some path $m \in F_r$ such that $ms^*s = s$, as required.

For the converse, let $r = 1$ and let $F_1$ be as in the statement. As it is finite, there is a finite $K_1 \subset K$ such that every element $m \in F_r$ is a word in $K_1$. Putting $c$ as the maximum length of the words in $F_1$, it then follows that $K_1$ and $c$ witness the FL condition of $(S, K)$. □

**Remark 5.2.19.** Proposition 5.2.18 gives a geometric interpretation of condition FL. Indeed, let $C_r$ be the set of points of $S$ such that $d(s^*s, s) \leq r$. Observe that if $S$ was a group then $C_r$ would be a *ball*, as there is only one $\mathcal{L}$-class. However, in general it is convenient to think of $C_r$ as a *cylinder*. Indeed, note that $C_r$ is a union of $r$-balls around every projection in $S$ (see Figure 5.2), forming a cylinder. Then, by Proposition 5.2.18, condition FL actually states that $C_r$ has a finite *cover*[1], meaning every point $s \in C_r$ is below some point in $F_r$.

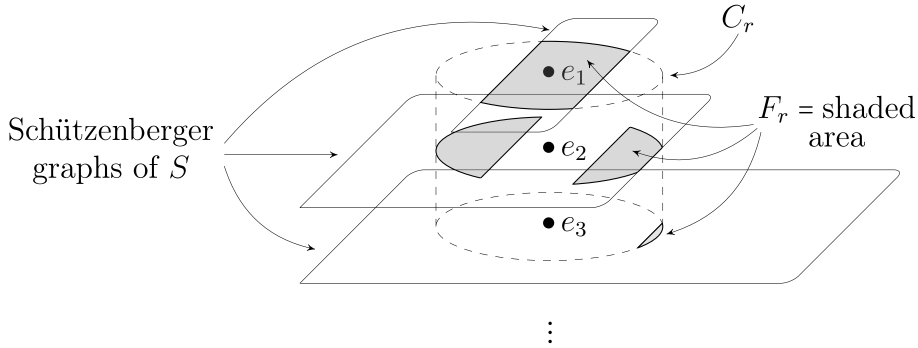


FIGURE 5.2: Rough depiction of the *finite labeleability* condition as stated in Proposition 5.2.18. In this sense, given a radius $r \geq 0$, the *cylinder* $C_r$ (dashed in the picture) is the set of points of $S$ such that $d(s^*s, s) \leq r$, and we view the finite set $F_r$ as a *cover* or a *tap* of $C_r$.

**Example 5.2.20.** Following Proposition 5.2.18 observe that the condition FL is trivially satisfied when $S = G$ is a countable group. Indeed, equipping $G$ with a proper and right invariant metric $d$ (see Proposition 2.2.17) yields a metric space

[1]Here we mean *cover* as in the geometric and intuitive idea, as in Figure 5.2. We do **not** mean *cover* as Exel does in [36].

$(G, d)$ of bounded geometry. Therefore the $r$-ball $B_r(1) \subset G$ is a finite set, and it trivially satisfies condition FL as stated in Proposition 5.2.18.

**Example 5.2.21.** A simple example of an inverse semigroup that does not admit finite labeling is $S = (\mathbb{N}, \min)$, where $n \cdot m := \min\{n, m\}$ and $n^* = n$. In this case, the generating set must necessarily be $K = S = E(S)$, and hence $\Lambda_S = \sqcup_{n \in \mathbb{N}}\{n\}$, i.e., $\Lambda_S$ is formed by infinitely many isolated points that are pairwise at infinite distance, where any vertex $\{n\}$ has infinitely loops labeled by $m \geq n$. It is clear that for any finite $K_1 \subset K$ there is an $n \in \mathbb{N}$ such that $n \geq k$ for any $k \in K_1$. Letting $x = y := n$ in Definition 5.2.13 it then follows that $S$ is not FL, even though any $\mathcal{L}$-class consisting of a single point is trivially FL. Similarly, the graph $\Lambda_T$ associated to the semigroup $T := (\mathbb{N}, \max)$ consists of infinitely many isolated points pairwise at infinite distance, and any vertex $\{n\}$ has precisely $n$ loops labeled by $k \in \{1, \ldots, n\}$. In this case, $\Lambda_T$ is FL because $T$ is F-inverse.

Note $S = (\mathbb{N}, \min)$ proves the converse implications in Proposition 5.2.17 are false, that is, $S$ does not have condition FL but satisfies both (1) and (2) in Proposition 5.2.17.

We turn to the proof of the main result, i.e., we want to compare the algebras $\mathcal{R}_S$ and $C_u^*(\Lambda_S)$. The following preliminary result shows that one inclusion always holds.

**Proposition 5.2.22.** *Let $S$ be an inverse semigroup. Let $\Lambda_S$ be the disjoint union of the left-Schützenberger graphs of $S$. Then $\mathcal{R}_S \subset C_u^*(\Lambda_S)$.*

*Proof.* Note first that any $f \in \ell^\infty(S)$ corresponds to a diagonal operator hence has propagation 0. Note that the propagation of the generators $V_s$, where $s \in S$, is given by

$$p(V_s) := \sup_{x \in D_{s^*s}} d(x, sx) = d(s^*s, s).$$

In fact, it is clear that $p(V_s) \geq d(s^*s, s)$ since $s^*s \in D_{s^*s}$. The reverse inequality follows from Lemma 5.1.7. □

**Theorem 5.2.23.** *Let $S = \langle K \rangle$ be a countable and discrete inverse semigroup such that $\Lambda_S$ is of bounded geometry. Then the following conditions are equivalent:*

*(1) $(S, K)$ admits a finite labeling (see Definition 5.2.13).*

*(2) The $*$-algebras $\mathcal{R}_{S,alg}$ and $C^*_{u,alg}(\Lambda_S)$ are equal, and hence*

$$\mathcal{R}_S = C_u^*(\Lambda_S).$$

*Proof.* (1) ⇒ (2). Given an operator $T \in C^*_{u,alg}(\Lambda_S)$, say of propagation $r > 0$, let $F_r$ be as in Proposition 5.2.18. Observe that for every pair $x, y \in S$ such that $d(x, y) \leq r$ there is $t \in F_r$ such that $tx = y$. Let $t_{x,y} \in F_r$ be such a possible choice and consider the functions $\xi_s \in \ell^\infty(S)$, $s \in S$, defined by

$$\xi_s(y) := \begin{cases} \langle \delta_y, T\delta_{s^*y}\rangle & \text{if } y\,\mathcal{L}\,s^*y \text{ and } s = t_{s^*y,y}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\xi_s = 0$ if $s \notin F_r$. We claim that

$$T = \sum_{s\in S} \xi_s V_s = \sum_{s\in F_r} \xi_s V_s.$$

Observe that, on the one hand, if $x$ and $y$ are not $\mathcal{L}$-related then $\langle \delta_y, T\delta_x\rangle = 0$ since $T$ is of bounded propagation. In addition, if $x \in D_{s^*s}$ then $x\mathcal{L}sx$ and, therefore, $sx \neq y$, which implies that $\langle \delta_y, (\sum_{s\in S} \xi_s V_s)\delta_x\rangle = 0$. On the other hand, if $x\mathcal{L}y$ then:

$$\left\langle \delta_y, \left(\sum_{s\in S} \xi_s V_s\right)\delta_x \right\rangle = \left\langle \delta_y, \sum_{\substack{s\in F_r \\ x\in D_{s^*s}}} \xi_s(sx)\,\delta_{sx} \right\rangle = \sum_{\substack{s\in F_r \\ sx=y}} \xi_s(y) = \langle \delta_y, T\delta_x\rangle,$$

since, by construction, there is exactly one $s \in F_r$ such that $sx = y$ and $\xi_s(y) \neq 0$, namely $s = t_{x,y}$.

(2) ⇒ (1). Suppose $(S,K)$ does not admit a finite labeling. Let $K = \{k_1, k_2, \ldots\}$ and denote by $K_n := \{k_1, \ldots, k_n\}$ be the first $n$ generators. Then for every $n \in \mathbb{N}$ there are points $x_n, y_n \in S$ such that $x_n\mathcal{L}y_n$, $y_n \in Kx_n$ and $x_n$ and $y_n$ are not joined by a path labeled by $K_n$ of length less than $n$. Note, as well, that since $\Lambda_S$ is of bounded geometry we may suppose that $x_n \neq x_{n'}$ for any $n \neq n'$. Consider the operator:

$$T\colon \ell^2 S \to \ell^2 S, \quad T\delta_z := \begin{cases} \delta_{y_n} & \text{if } z = x_n, \\ 0 & \text{otherwise.} \end{cases}$$

The operator $T$ has finite propagation since $\sup_{n\in\mathbb{N}} d(x_n, y_n) \leq 1$, and, thus, $T \in C^*_{u,\mathrm{alg}}(\Lambda_S)$. Moreover, we will show that $T$ cannot be approximated by elements in $\mathcal{R}_{S,alg}$ and, therefore, $T \notin \mathcal{R}_S$. Indeed, given any $\sum_{i=1}^m f_i V_{s_i} \in \mathcal{R}_{S,\mathrm{alg}}$, let $n \in \mathbb{N}$ be sufficiently large so that $s_i$ is a word in $K_n$ for every $i = 1, \ldots, m$ and $n$ is greater than the maximum length of the elements $s_i$. In this case

$$\begin{aligned} \left\| T - \sum_{i=1}^m f_i V_{s_i} \right\| &\geq \left\| T(\delta_{x_n}) - \left(\sum_{i=1}^m f_i V_{s_i}\right)(\delta_{x_n}) \right\|_{\ell^2(S)} \\ &= \left\| \delta_{y_n} - \sum_{\substack{i=1 \\ x_n\in D_{s_i^*s_i}}}^m f_i(s_i x_n)\,\delta_{s_i x_n} \right\|_{\ell^2(S)} \geq 1, \end{aligned}$$

where the last inequality follows from the fact that, by construction, $s_i x_n \neq y_n$ for all $i = 1, \ldots, m$. □

Note that, in general, the uniform Roe algebra decomposes as a direct sum over the algebras associated to the corresponding $\mathcal{L}$-classes

$$C_u^*(\Lambda_S) \cong \prod_{e \in E(S)} C_u^*(\Lambda_{L_e}).$$

Based on the strategy of the proof of the preceding theorem one can prove a similar result relating the uniform Roe algebra of an $\mathcal{L}$-class in $S$ and the corresponding corner of $\mathcal{R}_S$.

**Theorem 5.2.24.** *Let $S = \langle K \rangle$ be a countable and discrete inverse semigroup and let $L \subset S$ be an $\mathcal{L}$-class such that $\Lambda_L$ is of bounded geometry. Denote by $p_L$ be the orthogonal projection from $\ell^2 S$ onto $\ell^2 L \subset \ell^2 S$. Then the following conditions are equivalent:*

*(1) $(L, K)$ admits a finitely labeling (see Definition 5.2.13).*

*(2) $p_L \mathcal{R}_S p_L = C_u^*(\Lambda_L)$ as subalgebras of $\mathcal{B}(\ell^2 S)$.*

As a corollary of Theorem 5.2.23 and Proposition 5.2.15 we have the following:

**Corollary 5.2.25.** *Let $S$ be a finitely generated inverse semigroup or F-inverse semigroup such that $\Lambda_S$ has of bounded geometry. Then*

$$\mathcal{R}_S = C_u^*(\Lambda_S) = \ell^\infty(S) \rtimes_r S.$$

**Example 5.2.26.** Recall from Example 5.2.21 that the semigroup $S = (\mathbb{N}, \min)$ is not FL. Thus, by Theorem 5.2.23 we have $\mathcal{R}_S \neq C_u^*(\Lambda_S)$. Indeed, observe that $\Lambda_S = \sqcup_{n \in \mathbb{N}} \{n\}$ and, therefore, $C_u^*(\Lambda_S) = \ell^\infty(\mathbb{N})$. On the other hand, $\mathcal{R}_{S,\mathrm{alg}} = c_{\mathrm{fin}}(\mathbb{N})$ (sequences with finite support) and, hence, $\mathcal{R}_S = c_0(\mathbb{N})$.

## 5.3 Quasi-isometric invariants

In this section we study some of the *large scale properties* of the graph $\Lambda_S$ in relation to C*-properties of the reduced semigroup C*-algebra $C_r^*(S)$ and the uniform Roe algebra $C_u^*(\Lambda_S)$. Recall that a property $\mathfrak{P}$ is said to be a *quasi-isometric invariant* if, given two quasi-isometric extended metric spaces $(X, d_X)$ and $(Y, d_Y)$ such that $X$ has $\mathfrak{P}$, then so does $Y$. We will be particularly interested in two notions, namely *amenability* (see Section 5.3.1) and *property A* (see Section 5.3.2).

### 5.3.1 Domain-measurability (and not amenability)

In this subsection we show that domain measurability, as defined in Definition 4.1.5, is a quasi-isometric invariant of the graph $\Lambda_S$. We begin recalling that Følner sets may be localized within an $\mathcal{L}$-class of $S$. Indeed, observe that the following is a translation of Lemma 4.3.4 into our context.

**Lemma 5.3.1.** *Let $S$ be domain measurable. Then for every $\varepsilon > 0$ and finite $\mathcal{F} \subset S$, there exists some $\mathcal{L}$-class $L \subset S$ and a finite non-empty $F \subset L$ such that*

$$|s\,(F \cap D_{s^*s}) \cup F| \leq (1+\varepsilon)\,|F|$$

*for all $s \in \mathcal{F}$.*

The following two lemmas allow us to reduce the proof of Theorem 5.3.5 to the case of a surjective quasi-isometry.

**Lemma 5.3.2.** *Let $S,T$ be quasi-isometric semigroups of bounded geometry. Then there is a finite group $G$ and a surjective quasi-isometry $\varphi\colon S \times G \to T$.*

*Proof.* Let $\phi\colon S \to T$ be a quasi-isometry. In particular, there is some $r > 0$ such that $d_T(t, \phi(S)) \leq r$ for every $t \in T$. Let $m > 0$ be an upper-bound of the cardinality of the $r$-balls in $T$, and let $G$ be any finite group of cardinality $m$. For any $t \in T$ let $\theta_t$ be an embedding of $B_r(t)$ into $G$ and consider the map

$$\varphi\colon S \times G \to T, \ \text{ where } \varphi\,(s,g) := \begin{cases} \theta^{-1}_{\phi(s)}\,(g) & \text{if } g \in \operatorname{im}\big(\theta_{\phi(s)}\big) \\ \phi\,(s) & \quad\text{otherwise.} \end{cases}$$

Then $\varphi$ is surjective and a local perturbation of $\phi$ and, thus, a quasi-isometry. □

**Lemma 5.3.3.** *Let $S$ be an inverse semigroup, and let $G$ be a finite group. If $S \times G$ is domain measurable (respectively amenable) then so is $S$.*

*Proof.* Observe the product in $S \times G$ is defined coordinate-wise. In particular $(s,g)^*(s,g) = (s^*s, 1)$ and, therefore, $(x,g) \in D_{(s^*s,1)}$ if and only if $x \in D_{s^*s}$.

Given $\varepsilon > 0$ and a finite $\mathcal{F} \subset S$ let $F_G \subset S \times G$ witness the $(\varepsilon/|G|, \mathcal{F} \times G)$-Følner condition of $S \times G$. Consider the set

$$F := \{s \in S \mid (s,g) \in F_G \text{ for some } g \in G\}\,.$$

Then $F \subset S$ is clearly finite and non-empty. Furthermore $|F| \cdot |G| \geq |F \times G|$ and

$$\frac{|s\,(F \cap D_{s^*s}) \setminus F|}{|F|} \leq \frac{\big|(s,1)\,\big(F \times G \cap D_{(s^*s,1)}\big) \setminus F \times G\big|}{|F \times G|} \cdot |G| \leq \frac{\varepsilon}{|G|} \cdot |G| \leq \varepsilon.$$

for any $s \in \mathcal{F}$. Therefore $F$ witnesses the $(\varepsilon, \mathcal{F})$-Følner condition of $S$. If, in addition, $S \times G$ is amenable, then the set $F$ constructed before is contained in $D_{s^*s}$. □

The next proposition gives an alternative Følner type characterization of domain measurability in the case that $(S,K)$ admits a finite labeling. Recall that $\mathcal{N}_r A$ is the *$r$-neighborhood of $A$*, i.e., the set of points $x \in S$ such that $d(x,A) \leq r$. Note that, in particular, if $A \subset L$, where $L \subset S$ is an $\mathcal{L}$-class, then $A \subset \mathcal{N}_r A \subset L$.

**Proposition 5.3.4.** *Let $S = \langle K \rangle$ be an inverse semigroup admitting a finite labeling. Then $S$ is domain measurable if and only if for every $r > 0$ and $\varepsilon > 0$ there is a finite non-empty $F \subset L \subset S$, where $L$ is an $\mathcal{L}$-class, such that*

$$|\mathcal{N}_r F| \leq (1+\varepsilon)|F|.$$

*Proof.* It is standard to show that if $S$ is domain measurable then the Følner condition implies the corresponding inequality of the $r$-neighborhood. To show the reverse implication let $F_r$ be as in Proposition 5.2.18. By Lemma 5.3.1 there is an $\mathcal{L}$-class $L$ and a finite $F \subset L$ such that for every $s \in F_r$

$$\frac{|s\,(F \cap D_{s^*s}) \cup F|}{|F|} \leq 1 + \frac{\varepsilon}{|F_r|}.$$

It follows that such an $F$ satisfies the required Følner condition. □

**Theorem 5.3.5.** *Let $S$ and $T$ be inverse semigroups admitting finite labelings, and suppose $S$ and $T$ are quasi-isometric. If $T$ is domain measurable then so is $S$.*

*Proof.* Let $\varphi\colon S \to T$ be a quasi-isometry. By Lemma 5.3.2 and Lemma 5.3.3 we may suppose that $\varphi$ is a surjective quasi-isometry. Let $m > 0$ and $c \geq 0$ be some constants such that for every $x, y \in S$

$$\frac{1}{l} d_S\,(x,y) - c \leq d_T\,(\varphi\,(x)\,,\varphi\,(y)) \,\leq\, l\, d_S\,(x,y) + c,$$

where $d_S$ and $d_T$ denote the path distances in $\Lambda_S$ and $\Lambda_T$ respectively. By Proposition 5.3.4 and given $r > 0$ and $\varepsilon > 0$ there is some finite non-empty $F_T \subset T$ within some $\mathcal{L}$-class and with small $(lr + c)$-neighborhood, that is, $|\mathcal{N}_{lr+c} F_T| \leq (1+\varepsilon)|F_T|$. We claim the set

$$F_S := \varphi^{-1}\,(F_T)$$

has a small $r$-neighborhood in $S$ and, therefore, by Proposition 5.3.4, $S$ is domain measurable too. Observe that $F_S$ is non-empty since $\varphi$ is surjective. Furthermore, the distance between two any $\mathcal{L}$-classes (in either $\Lambda_S$ or $\Lambda_T$) is infinite, and thus, since $\varphi$ is a quasi-isometry, it takes $\mathcal{L}$-classes of $S$ onto $\mathcal{L}$-classes of $T$. $F_S$ is, whence, contained within some $\mathcal{L}$-class. Moreover, $F_S$ must be finite since $T$ is of bounded geometry and $\varphi$ is a quasi-isometry. Finally:

$$\frac{|\mathcal{N}_r F_S|}{|F_S|} \leq \frac{|\mathcal{N}_{lr+c} F_T|}{|F_T|} \leq 1 + \varepsilon,$$

which proves that $F_S$ has small $r$-neighborhood in $S$. □

Figure 5.3 illustrates the main ideas of the proof of Theorem 5.3.5.

**Remark 5.3.6.** Note that Theorem 5.3.5 only proves that domain measurability is a quasi-isometry invariant. Indeed, following our approach Day's amenability is *not* a

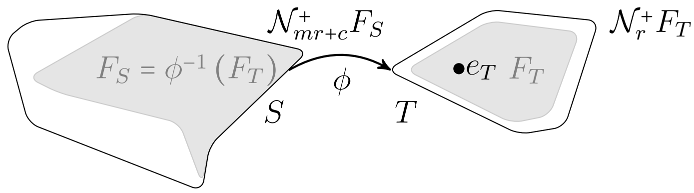


FIGURE 5.3: Graphical depiction of the fact that domain-measurability is a quasi-isometric invariant. The notation is the same as the one used in the proof of Theorem 5.3.5.

quasi-isometric invariant, since the quasi-isometry $\phi\colon S \to T$ need not respect the $\mathcal{L}$-classes and, therefore, need not respect the localization condition (see Theorem 4.1.3 and Definition 4.1.5). For instance, letting $S := \{1\} \sqcup \mathbb{F}_2$ and $T := \mathbb{F}_2 \sqcup \{0\}$, then $T$ is amenable, $S$ is not amenable, and the map

$$\varphi\colon S \to T, \quad \text{where } \varphi(1) = 0 \text{ and } \varphi|_{\mathbb{F}_2} = id_{\mathbb{F}_2}$$

is a quasi-isometry, since 1 and $\mathbb{F}_2$ are in different connected components of $S$, and so are $\mathbb{F}_2$ and 0 in $T$.

### 5.3.2 Property A

We now address another important coarse invariant of metric spaces of bounded geometry, namely Yu's property A (see Definition 2.2.12). As it turns out we can characterize when the Schützenberger graph $\Lambda_L$ of an $\mathcal{L}$-class $L \subset S$ has property A, generalizing the result for groups (see Theorem 2.2.21).

**Theorem 5.3.7.** *Let $S$ be a countable and discrete inverse semigroup, and let $L \subset S$ be an $\mathcal{L}$-class such that the left Schützenberger graph $\Lambda_L$ is of bounded geometry. Let $p_L$ be the orthogonal projection from $\ell^2 S$ onto $\ell^2 L \subset \ell^2 S$. Consider the following statements:*

*(1) The graph $\Lambda_L$ has property A.*
*(2) The C*-algebra $p_L \mathcal{R}_S p_L$ is nuclear.*
*(3) The C*-algebra $p_L \mathcal{R}_S p_L$ is exact.*
*(4) The C*-algebra $\overline{p_L C^*_r(S) p_L}$ is exact.*

*Then (1) ⇒ (2) ⇒ (3) ⇔ (4). Moreover, if $(L, K)$ admits a finite labeling then all conditions are equivalent.*

*Proof.* For convenience, and since $L$ is fixed, in this proof we denote by $p$ the projection $p_L$, and $\mathcal{R}$ stands for $\mathcal{R}_S$. Furthermore, note that $pV_s p = V_s p$, since $p$ projects onto an $\mathcal{L}$-class (see Lemma 4.3.3).

(1) ⇒ (2). Let $\mathcal{F} \subset p\mathcal{R}p$ be finite and $\varepsilon > 0$. Observe that, without loss of generality, we may suppose that every element of $\mathcal{F}$ is of the form $fV_s$ where the

support of $f$ is contained in $L \cap D_{ss^*}$. Let $r > 0$ be larger than the length of $s$ for every $s$ such that $fV_s \in \mathcal{F}$. Since $\Lambda_L$ has property A there are $c > 0$ and $\xi \colon \Lambda_L \to \ell^1(\Lambda_L)_{1,+}$ such that $\operatorname{supp}(\xi_x) \subset B_c(x)$, $x \in L$, and

$$\| \xi_x - \xi_y \|_1 \leq \frac{\varepsilon}{m} \text{ for every } x, y \in L \text{ such that } d(x,y) \leq r, \tag{5.3}$$

where $m := \max\{ \|fV_s\| \mid fV_s \in \mathcal{F} \}$. Consider then the map

$$\varphi \colon p\mathcal{R}p \to \prod_{x \in L} M_{B_C(x)}, \text{ where } \varphi(a) := \left(p_{B_c(x)} a p_{B_c(x)}\right)_{x \in L}.$$

Recall that $M_{B_c(x)}$ denotes the full matrix algebra with rows and columns labeled by elements in $B_c(x)$ and $p_{B_c(x)} \in \ell^\infty(S)$ is the characteristic function of $B_c(x)$. Since $\Lambda_L$ is of bounded geometry there is some $q > 0$ such that $|B_R(x)| \leq q$ for every $x \in L$ and, therefore,

$$\operatorname{im}(\varphi) \subset \prod_{x \in L} M_{B_C(x)} \subset \ell^\infty(L) \otimes M_q.$$

Let $A$ be the closure in norm of the image of $\varphi$ and consider $\varphi \colon p\mathcal{R}p \to A$. Moreover, let

$$\psi \colon A \to p\mathcal{R}p, \text{ where } \psi\left((b_x)_{x \in L}\right) := \sum_{x \in L} \xi_x^* b_x \xi_x$$

where we identify $\xi_x$ with the diagonal operator $\xi_x \delta_y := \xi_y(x)\delta_y$ (note the flip between the argument and the index of $\xi$). It is clear that both $\varphi$ and $\psi$ are c.c.p. maps. Furthermore

$$\psi(\varphi(fV_s)) = \sum_{x \in L} f\xi_x V_s \xi_x = fV_s \left( \sum_{x \in L} (s^*\xi_x)\, \xi_x \right) \in p\mathcal{R}p,$$

where $s^*\xi_x(\delta_y) := \xi_{sy}(x)$ if $y \in D_{s^*s}$ and $s^*\xi_x(\delta_y) = 0$ otherwise. From Eq. (5.3) we have for all $s \in S$ such that $fV_s \in \mathcal{F}$ the following estimate

$$\begin{aligned}\left\| V_{s^*s} - \sum_{x \in L} (s^*\xi_x)\, \xi_x \right\| &= \sup_{y \in L} \left| V_{s^*s}\delta_y - \sum_{x \in L} \left((s^*\xi_x)\, \xi_x\right) \delta_y \right| \\ &= \sup_{y \in L \cap D_{s^*s}} \left| 1 - \sum_{x \in L} \xi_y(x)\, \xi_{sy}(x) \right| = \sup_{y \in L \cap D_{s^*s}} |1 - \langle \xi_{sy}, \xi_y \rangle| \leq \frac{\varepsilon}{M}.\end{aligned}$$

Therefore, we can estimate

$$\begin{aligned}\| fV_s - \psi(\varphi(fV_s)) \| &= \left\| fV_s \left( V_{s^*s} - \sum_{x \in L} (s^*\xi_x)\, \xi_x \right) \right\| \\ &\leq \|fV_s\| \left\| V_{s^*s} - \sum_{x \in L} (s^*\xi_x)\, \xi_x \right\| \leq \varepsilon.\end{aligned}$$

Since, by Proposition 2.3.2, $A$ is nuclear it follows that $p\mathcal{R}p$ is nuclear as well by Proposition 2.3.3.

(2) ⇒ (3) follows since nuclear algebras are exact.

(3) ⇒ (4) follows since exactness passes to subalgebras.

(4) ⇒ (3). Given $\varepsilon > 0$ and a finite $\mathcal{F} \subset p\mathcal{R}p$, without loss of generality we may suppose that every element in $\mathcal{F}$ is of the form $fV_s$, where the support of $f$ is contained in $L \cap D_{ss^*}$. By the exactness of $\overline{pC_r^*(S)p}$ there are c.c.p. maps

$$\tilde{\varphi}\colon \overline{pC_r^*(S)p} \to M_n \text{ and } \tilde{\psi}\colon M_n \to \mathcal{B}\left(\ell^2 L\right)$$

such that $\|V_s p - \tilde{\psi}(\tilde{\varphi}(V_s p))\| \leq \varepsilon/m$ for all $fV_s \in \mathcal{F}$, where $m := \max\{\|fV_s\|\}_{fV_s \in \mathcal{F}}$. Consider the c.c.p. maps

$$\begin{aligned} \varphi\colon p\mathcal{R}p &\to \ell^\infty(L) \otimes M_n \quad \text{and} \quad \psi\colon \ell^\infty(L) \otimes M_n \to \mathcal{B}\left(\ell^2(L)\right), \\ fV_s &\to f \otimes \tilde{\varphi}(V_s) \quad \text{and} \quad f \otimes b \to f\tilde{\psi}(b). \end{aligned}$$

Then

$$\|\psi(\varphi(fV_s)) - fV_s\| \leq \|fV_s\| \cdot \left\|V_s p - \tilde{\psi}(\tilde{\varphi}(V_s p))\right\| \leq \varepsilon,$$

which proves, again using that $\ell^\infty(L) \otimes M_n$ is nuclear (see Proposition 2.3.2), that $p\mathcal{R}p$ is exact.

Finally, if the $\mathcal{L}$-class $L$ admits a finite labeling then by Theorem 5.2.24 the corner $p\mathcal{R}p$ is a uniform Roe algebra, i.e., $p\mathcal{R}p \cong C_u^*(\Lambda_L)$. Using recent results by Sato (see [94, Theorem 1.1]) we have that $C_u^*(\Lambda_L)$ is exact ⇔ $C_u^*(\Lambda_L)$ is nuclear ⇔ $\Lambda_L$ has property A and, therefore, all the statements in the theorem are equivalent. □

**Remark 5.3.8.** Observe that implication of (1) ⇒ (2) in Theorem 5.3.7 is independent of $(L, K)$ admitting a finite labeling or not. This is remarkable since, in general, nuclearity does not pass to subalgebras and the corner $p_L\mathcal{R}_S p_L$ is properly contained in the uniform Roe algebra $C_u^*(\Lambda_L)$. In general, it is known that a metric space has property A if and only if its uniform Roe algebra is nuclear (see, for example, [20, Theorem 5.5.7]).

Moreover, observe that in Theorem 5.3.7 (4) the corner $p_L C_r^*(S) p_L$ need not be closed. Indeed, note that, in general, $p_L$ is not contained in $C_r^*(S)$.

We can also extend Theorem 5.3.7 to the whole graph $\Lambda_S = \sqcup_{e \in E(S)} \Lambda_{L_e}$. In particular, this allows us to characterize when the graph $\Lambda_S$ has property A via the C*-algebras $\mathcal{R}_S$ and $C_r^*(S)$. We first note that $\Lambda_S$ has property A when all of its connected components have uniform property A (see Definition 2.2.12).

**Lemma 5.3.9.** *$\Lambda_S$ has property A if and only if the family $\{\Lambda_{L_e}\}_{e \in E(S)}$ has uniform property A.*

*Proof.* The proof is a direct consequence of the definitions involved. □

Observe that there are inverse semigroups, necessarily with infinitely many $\mathcal{L}$-classes, without property A and such that every of their Schützenberger graphs do have property A (see Section 5.4.5).

**Theorem 5.3.10.** *Let $S$ be a countable and discrete inverse semigroup of bounded geometry. Consider the assertions:*

*(1) The graph $\Lambda_S$ has property A.*
*(2) The C*-algebra $\mathcal{R}_S$ is nuclear.*
*(3) The C*-algebra $\mathcal{R}_S$ is exact.*
*(4) The C*-algebra $C_r^*(S)$ is exact.*

*Then (1) ⇒ (2) ⇒ (3) ⇔ (4). Moreover, if $S$ admits a finite labeling then all conditions are equivalent.*

*Proof.* By Lemma 5.3.9, condition (1) is equivalent to the family $\{\Lambda_{L_e}\}_{e\in E(S)}$ having uniform property A. Observe that all the implications follow from similar arguments as those in the proof of Theorem 5.3.7. For instance, in this context the c.c.p. maps are given for each $\mathcal{L}$-class $L_e$ and one needs to use the uniform bound on the constants $c_e$, $e \in E$ (see Definition 2.2.12). □

We end the section giving a relation between the property A of $S$ and that of $G(S)$. Observe that the proof of the following proposition is based on the facts that the left Cayley graph of $G(S)$ is the inductive limit of the Schützenberger graphs of $S$ and that property A is closed under inductive limits with injective connecting maps. We give an explicit proof because, in general, the natural connecting maps $L_e \to L_{ef}$, where $se \mapsto sef$, need not be injective for certain $\mathcal{L}$-classes.

**Proposition 5.3.11.** *Let $S$ be a countable and discrete inverse semigroup such that $S$ is of bounded geometry. If $S$ has property A, then so does the maximal homomorphic image $G(S)$.*

*Proof.* Fix a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ and let $\{e_n\}_{n\in\mathbb{N}} \subset E(S)$ be a decreasing sequence of projections that is eventually below every other projection (as in Lemma 4.2.17). In particular, observe that for all $s \in S$ we have that $se_n\mathcal{L}e_n$ for all $n \geq n_0$ and $n_0 \in \mathbb{N}$ large enough. Let $L_n \subset S$ be the $\mathcal{L}$-class of $e_n \in E(S)$. By Lemma 5.3.9 the family $\{\Lambda_{L_n}\}_{n\in\mathbb{N}}$ has uniform property A and, thus, given $\varepsilon > 0$ and $r > 0$ let $c \leq 0$ and $\xi^{(n)}\colon \Lambda_{L_n} \to \ell^1(\Lambda_{L_n})_{1,+}$ witness the $(r,\varepsilon)$-uniform property A of the family $\{\Lambda_{L_n}\}_{n\in\mathbb{N}}$. Consider

$$\xi\colon G(S) \to \ell^1(G(S)), \quad \text{where } \xi_{\sigma(s)}(\sigma(t)) := \lim_{n\to\omega} \xi^{(n)}_{se_n}(te_n).$$

Note that, since $se_n\mathcal{L}e_n$ for all $n \geq n_0$, the function $\xi_{\sigma(s)}$ is well defined. Furthermore, $\xi_{\sigma(s)}$ is positive and of norm 1, since

$$\left\|\xi_{\sigma(s)}\right\|_1 = \sum_{\sigma(t)\in G(S)} \lim_{n\to\omega} \xi^{(n)}_{se_n}(te_n) = \lim_{n\to\omega} \sum_{t\in B_c(se_n)} \xi^{(n)}_{se_n}(te_n) = \lim_{n\to\omega} \left\|\xi^{(n)}_{se_n}\right\|_1 = 1.$$

In addition, for any $\sigma(s), \sigma(t) \in G(S)$ such that $\xi_{\sigma(s)}(\sigma(t)) \neq 0$, by the limit construction it follows that there is some positive $\delta > 0$ with $\xi_{se_n}^{(n)}(te_n) \geq \delta$ for all sufficiently large $n \in \mathbb{N}$. Therefore, using the comparison of distances given in Lemma 5.1.7 we obtain

$$d_{G(S)}\left(\sigma\left(s\right), \sigma\left(t\right)\right) \leq d_{L_n}\left(se_n, te_n\right) \leq c < \infty$$

where $c > 0$ is a constant bounding the diameter of the supports of $\xi_{se_n}^{(n)}$. It thus follows that $\operatorname{supp}(\xi_{\sigma(s)})$ is contained in a ball of radius $c > 0$ around $\sigma(s)$. Finally, let $\sigma(s), \sigma(t) \in G(S)$ such that $d_{G(S)}(\sigma(s), \sigma(t)) \leq r$. Then

$$\left\|\xi_{\sigma(s)} - \xi_{\sigma(t)}\right\|_1 = \lim_{n \to \omega} \|\xi_{se_n} - \xi_{te_n}\|_1 \leq \varepsilon$$

since $\|\xi_{se_n} - \xi_{te_n}\|_1 \leq \varepsilon$ for all sufficiently large $n \in \mathbb{N}$. □

### 5.3.3 Property A in the E-unitary case

We conclude the study of property A by applying Theorem 5.3.10 to the special case where $S$ is E-unitary. The following proposition is an improvement of Proposition 5.2.2 in the E-unitary case.

**Proposition 5.3.12.** *Let $S$ be an E-unitary inverse semigroup, and let $G(S)$ be its maximal homomorphic image. Then the following are equivalent:*

*(1) $\Lambda_S$ is of bounded geometry.*

*(2) The left Cayley graph of $G(S)$ is of bounded geometry.*

*Proof.* The implication (1) ⇒ (2) was already proved in Proposition 5.2.2. To show the reverse implication let $r > 0$ and note that, by hypothesis, the $r$-balls of $G(S)$ have uniformly bounded cardinality, i.e., $\sup_{x \in S} |B_r(\sigma(x))| < \infty$. Since $S$ is E-unitary the canonical projection $\sigma$ gives an injective map from $B_r(x) \subset S$ to $B_r(\sigma(x)) \subset G(S)$ for every $x \in S$ and, thus,

$$\sup_{x \in S} |B_r\left(x\right)| \leq \sup_{x \in S} |B_r\left(\sigma\left(x\right)\right)| < \infty,$$

which proves that $S$ is of bounded geometry. □

Observe Proposition 5.3.12 actually proves that any upper bound on the cardinality of the $r$-balls of $S$ is also an upper bound on the cardinality of the $r$-balls of $G(S)$. The reverse implication is also true if $S$ is, in addition, E-unitary. Moreover, note that the E-unitary assumption on $S$ is essential in Proposition 5.3.12. Indeed, as an example of a non-E-unitary semigroup that fails to have this property let $S := G \sqcup \{0\}$, where 0 denotes a zero element and $G$ is a non-finitely generated group. Then $G(S)$ is trivial (hence of bounded geometry), but the left Schützenberger graph of the $\mathcal{L}$-class of $1_G$ is the left Cayley graph of $G$ and, thus, not of bounded geometry.

Recall that it was proven in [2, Proposition 8.5] that an E-unitary inverse semigroup $S$ is exact (in the sense that $C_r^*(S)$ is an exact C*-algebra) if and only if $G(S)$ is an exact group. With the techniques presented previously we can give a new proof of this result relating exactness of the C*-algebra to the property A of the graph $\Lambda_S$.

**Theorem 5.3.13.** *Let $S$ be an E-unitary countable and discrete inverse semigroup admitting a finite labeling. Then the following are equivalent:*

*(1) The graph $\Lambda_S$ has property A.*

*(2) The C*-algebra $C_r^*(S)$ is exact.*

*(3) The maximal homomorphic image $G(S)$ has property A.*

*Proof.* For the equivalence (1) $\Leftrightarrow$ (2) see Theorem 5.3.10. The implication (1) $\Rightarrow$ (3) is proved in Proposition 5.3.11, and, thus, it suffices to show (3) $\Rightarrow$ (2). It is well known (see, e.g., [3]) that $C_r^*(S) = C_0(\hat{E}_0) \rtimes_r G(S)$, where $\hat{E}_0$ is the spectrum of $S$. Thus, following similar reasonings as in the proof of the same implication of Theorems 5.3.7 and 5.3.10, it follows that if $G(S)$ is exact then so is $C_r^*(S)$. □

## 5.4 Examples of metric spaces coming from semigroups

This section aims to give several examples of metric spaces arising from inverse semigroups. The examples we introduce here are *almost orthogonal to*, meaning *have small intersection with*, the examples of Chapter 4 (see Section 4.4). In particular, the examples here considered aim to study the possible geometries that arise from inverse semigroups, and study these in relation with notions from this chapter, such as admitting a finite labeling. As such, Subsection 5.4.1 studies the known Schützenbergerg graphs of the polycyclic monoids, while Subsection 5.4.2 recalls a usual construction in semigroup literature. Subsection 5.4.3 then proves the well known fact that every undirected graph arises as the Schützenberger graph of an inverse semigroup, and Subsection 5.4.4 sketches an example of an E-unitary semigroup whose canonical projection $\sigma\colon S \to G(S)$ does not coarsely embed some $\mathcal{L}$-classes into $G(S)$. Lastly, Subsection 5.4.5 replicates certain interesting behavior in the context of inverse semigroups.

### 5.4.1 Polycyclic monoids

Given $n \in \mathbb{N}$ recall the construction of the polycyclic monoid $\mathcal{P}_n$ (see Example 3.3.3). It is finitely generated and, therefore, admits a finite labeling. Moreover, every non-trivial left Schützenberger graph of $\mathcal{P}_n$ is the $n$-ary complete rooted tree (see Figure 5.4 for $n = 2$ and Figure 5.1 for $n = 1$).

$$\mathcal{P}_2 = \langle 1, 0, a, b \mid a^*a = b^*b = 1 \text{ and } a^*b = 0 \rangle$$

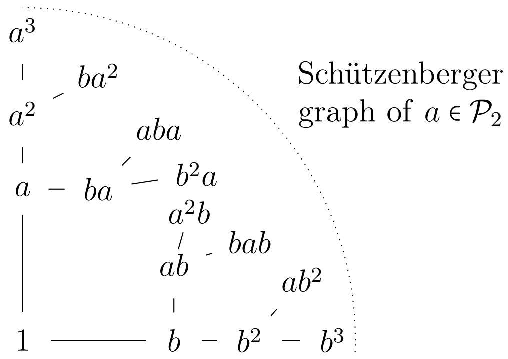


FIGURE 5.4: Schützenberger graph of $a$ in the polycyclic monoid $\mathcal{P}_2$.

### 5.4.2 Lattices and semidirect products

An important class of inverse semigroups arises from *lattices* and *semi-direct products*. We first recall these notions. Let $(X, \wedge)$ be a meet-semilattice, that is, let $X$ be a partially ordered set such that for every $x, y \in X$ there is a unique greatest lower bound of $x, y$, which we denote by $x \wedge y$. Suppose furthermore that a discrete group $G$ acts on $X$ by order-preserving bijections, and consider the inverse semigroup $X \rtimes G$ given (as a set) by $X \times G$. For $(x, g), (y, h) \in X \rtimes G$ we define their product to be

$$(x, g) \cdot (y, h) := (x \wedge gy, gh) \text{ and } (x, g)^* = \left(g^{-1}x, g^{-1}\right).$$

One can then show that $(X \rtimes G, \cdot)$ is an inverse semigroup, whose set of idempotents is canonically isomorphic to $X$. The following straightforward result characterizes when these semigroups admit finite labelings.

**Proposition 5.4.1.** *Let $X$ be a meet-semilattice, and let $G$ be a group acting on it. Let $X \rtimes G$ be constructed as above. Then the semigroup $X \rtimes G$ admits a finite labeling if, and only if, $G$ is finitely generated (as a group) and $X$ has finitely many maximal elements that bound above every other element in $X$.*

*Proof.* First suppose that $G = \langle K \rangle$, where $K$ is finite, and that $\{m_i\}_{i \in 1}^k \subset X$ are maximal points bounding above every element in $X$. Then for any radius $r > 0$ the set

$$F_r := \{(m_i, g) \text{ where } i = 1, \ldots, k \text{ and } g \in B_r(1) \subset G\}$$

witnesses the FL condition of $X \rtimes G$ as stated in Proposition 5.2.18. Indeed, $F_r$ is obviously finite, and if $(x, h) \in X \rtimes G$ is such that

$$r \geq d\left((x, h)^* (x, h), (x, h)\right) = d\left(\left(h^{-1}x, 1\right), (x, h)\right) = d_G(1, h)$$

then $h \in B_r(1_G) \subset G$. Moreover, as $x \in X$ there is some $i_0 \in \{1, \ldots, k\}$ such that $m_{i_0} \geq x$. Therefore the element $(m_{i_0}, h) \in F_r$ and $(m_{i_0}, h) \geq (x, h)$, proving the claim.

Now suppose $X \rtimes G$ admits a finite labeling, and let $F_1$ be as in Proposition 5.2.18 for $r = 1$. Then it is routine to show there are finite sets $K \subset G$ and $M \subset X$ such that $F_1 \subset M \times K$. Moreover, $K$ then generates $G$, while $M$ bounds above every element in $X$. $\square$

Particularly simple cases of $X \rtimes G$ come from considering $G$ to be trivial, i.e., $X$ equipped with the operation given by $x \cdot y := x \wedge y$. Several examples of such semi-lattices are the following (see these and Figure 5.5):

1. $S_1 := (\mathbb{N}, n \cdot m = \max\{n, m\})$. Since $S_1$ is F-inverse, all versions of the Roe algebra coincide by Corollary 5.2.25. In this case, $\mathcal{R}_{S_1} = \ell^\infty(\mathbb{N})$.
2. $S_2 := (\mathbb{N}, n \cdot m = \min\{n, m\})$. Observe $S_2$ has no maximal element, and hence $\mathrm{C}^*_u(\Lambda_{S_2}) = \ell^\infty(\mathbb{N}) \neq c_0(\mathbb{N}) = \mathcal{R}_{S_2}$.
3. $S_3 := \mathbb{N}$, where $n \cdot m = 0$ if $n \neq m$ and $n \cdot n = n$. $S_3$ has infinitely many maximal elements, and therefore $\mathrm{C}^*_u(\Lambda_{S_3}) = \ell^\infty(\mathbb{N}) \neq c_0(\mathbb{N}) = \mathcal{R}_{S_3}$.

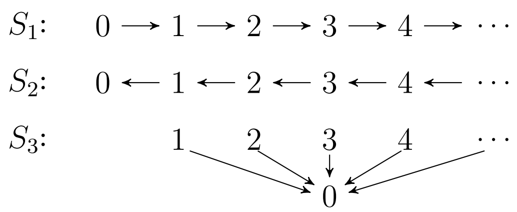


FIGURE 5.5: Graph representation of the semigroups $S_1, S_2$ and $S_3$. The arrows represent the partial order $\geq$.

### 5.4.3 Graphs arising as Schützenberger graphs

It is well known that not every $K$-labeled graph of bounded geometry is the left Cayley graph of a group generated by the set $K$. For example, the connected half-line $\mathbb{N}$ is clearly not the Cayley graph of any group. The following proposition shows that a large class of graphs can be realized as Schützenberger graphs.[2]

**Proposition 5.4.2.** *Let $\mathcal{G} = (V, E)$ be a non-empty, connected and undirected graph without multiple edges and suppose that every vertex is connected with itself via a loop. Then there is an inverse semigroup $S$ and an $\mathcal{L}$-class $L \subset S$ such that $\Lambda_L$ and $\mathcal{G}$ are isomorphic as graphs.*

*Proof.* We will only sketch the main ideas of the proof (see also [105, 66]). It is useful for the construction to think of the undirected edges of $\mathcal{G}$ as a pair of edges

[2] I would like to thank Nóra Szakács for pointing out the proof of Proposition 5.4.2.

with opposite orientations. We will denote these as $E \sqcup E^*$, where $(v_2, v_1)^* = (v_1, v_2)$. Fix an arbitrary vertex $v_0 \in V$ and consider the set of cycles in $\mathcal{G}$ starting at $v_0$:

$$\mathcal{C}(v_0) := \{ (v_0, v_p) \dots (v_2, v_1)(v_1, v_0) \text{ such that } (v_{i+1}, v_i), (v_0, v_p) \in E \sqcup E^* \} .$$

Consider the semigroup $S$ generated by $V \sqcup E \sqcup E^* \sqcup \{0\}$ and with relations:

- $v = v^2 = v^* = (v, v)$ for all $v \in V$.
- $v_2(v_2, v_1) = (v_2, v_1) = (v_2, v_1)v_1$ for all edges $(v_2, v_1) \in E$.
- $v_3(v_2, v_1) = 0$ if $v_2 \neq v_3$.
- $(v_2, v_1)v_3 = 0$ if $v_1 \neq v_3$.
- $\omega = v_0^* v_0$ for all $\omega \in \mathcal{C}(v_0)$.

Then, taking $K := V \sqcup E \sqcup E^* \sqcup \{0\}$, the left Schützenberger graph of the $\mathcal{L}$-class of $v_0$ can naturally be seen as oriented paths in $\mathcal{G}$ starting at $v_0$. Indeed, note that non-zero elements in $S$ are formal expressions $\mathfrak{p} = (v_p, v_{p-1}) \dots (v_2, v_1)$, where $(v_i, v_{i-1})$ are edges in $\mathcal{G}$, and $\mathfrak{p}\mathcal{L}v_0$ if and only if $\mathfrak{p}$ starts at $v_0$. Moreover, if $\mathfrak{p}, \mathfrak{q}$ are two paths in $\mathcal{G}$ with the same initial vertex $v$ and final vertex $w$, it follows that $\mathfrak{q}^*\mathfrak{p} = v$ and $\mathfrak{p}\mathfrak{q}^* = w$. Therefore

$$\mathfrak{p} = \mathfrak{p}v = \mathfrak{p}\mathfrak{q}^*\mathfrak{p} \text{ and } \mathfrak{q}^* = \mathfrak{q}^* w = \mathfrak{q}^*\mathfrak{p}\mathfrak{q}^*,$$

which implies that $\mathfrak{p} = \mathfrak{q}$. Hence the map $\mathfrak{p} \mapsto r(\mathfrak{p})$, sending each path $\mathfrak{p}$ to its final vertex is a natural bijection between the $\mathcal{L}$-class of $v_0$ in $S$ and the graph $\mathcal{G}$. Observe, as well, that two elements $\mathfrak{p}, \mathfrak{q} \in S$ are joined by an edge in $\Lambda_S$ if one is a prefix of the other, and therefore the map above is a graph isomorphism. □

**Remark 5.4.3.** The construction of $S$ in the proof of the preceding proposition can be also done in the more general case where the graph $\mathcal{G} = (V, E)$ is $K$-labeled with $K$ a set, as long as the labeling of the graph is *deterministic* (see [66]). Moreover, in that case the graph isomorphism respects the $K$-labeling.

Note, as well, that we require the vertices of $\mathcal{G}$ to be decorated with a loop. This condition is irrelevant from a large-scale geometry point of view and, therefore, we have the following straightforward corollary.

**Corollary 5.4.4.** *Any simple, connected graph can be quasi-isometrically realized as a Schützenberger graph of an inverse semigroup.*

For particular examples see Figure 5.6.

### 5.4.4 $\sigma$ is not a quasi-isometric embedding

This section recalls a, maybe shocking, example.[3] Let $S$ be an E-unitary inverse semigroup and, to simplify notation, denote $G(S)$ simply by $G$. We know that the canonical projection $\sigma\colon S \to G$ is injective when restricted to the $\mathcal{L}$-classes of $S$ (see

[3] The author is grateful to Benjamin Steinberg for pointing to it.

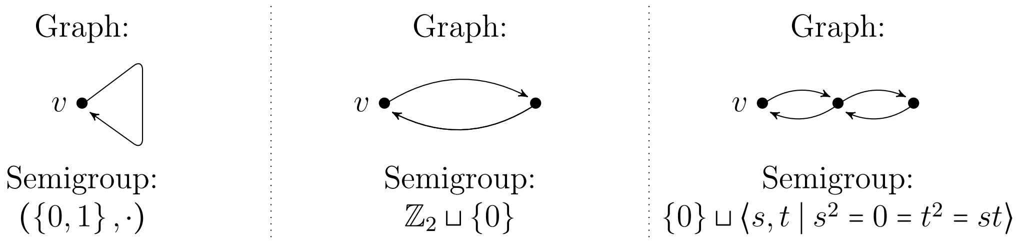


FIGURE 5.6: Some (undirected and connected) graphs and their associated inverse semigroups, following the construction of Proposition 5.4.2. In particular, the Schützenberger graph of $v$ in each semigroup is the correspondent graph.

Proposition 5.2.7). Moreover, from Lemma 5.1.7 we also know that $d_G(\sigma(x),\sigma(y)) \leq d_S(x,y)$ and, therefore, it is only natural to wonder the following:

*Does $\sigma$ quasi-isometrically embed every Schützenberger graph of $S$ into $G$?*

In even greater generality, one could wonder if:

*Does $\sigma$ coarsely embed every Schützenberger graph of $S$ into $G$?*

Indeed, note that, a priori, it could happen that $\sigma$ is a coarse embedding and not a quasi-isometry. An instance of this behavior is the fact that if $H$ is a subgroup of $G$ then the inclusion of $H$ into $G$ need not be a quasi-isometry, but it always is a coarse embedding (see [76, Example 1.4.8]). The answer to the second question (and, hence, also to the first) is negative.

Given a finitely generated group $G$, let $S$ be the *Meakin-Margolis expansion of* $G$, that is, $S$ is the set of pairs $(X,g)$, where $g \in G$ and $X$ is a finite connected subgraph of the Cayley graph of $G$ containing both $1 \in G$ and $g$. We equip $S$ with the product given by

$$(X,g)\cdot(Y,h) := (X \cup gY, gh),$$

where $\cup$ means the union of graphs as subgraphs of $G$. This process yields an inverse semigroup, whose product is illustrated in Figure 5.7. For instance, observe that $(X,g)^* = (g^{-1}X, g^{-1})$, and that $(X,g) \in E(S)$ if and only if $g = 1$. Moreover, the partial order of $S$ is given by

$$(X,g) \geq (Y,h) \Leftrightarrow g = h \text{ and } X \subset Y,$$

while $(X,g)$ and $(Y,h)$ are $\sigma$-related if, and only if, $g = h$. Note that, therefore, the projection onto $G$ is exactly the canonical quotient map $\sigma$, and hence $S$ is always E-unitary. The following characterizes when $S$ is F-inverse.

**Proposition 5.4.5.** *Let $G$ be a finitely generated group, and let $S$ be the Meakin-Margolis extension of $G$. Then $S$ is F-inverse if, and only if, $G$ is free.*

*Proof.* Given $(X,g)\in S$ observe that the maximal elements of the $\sigma$-class of $(X,g)$ are of the form $(\omega,g)$, where $\omega\subset G$ is a simple path from 1 to $g$.

Now, if $G$ is non-free, then there is a non-trivial cycle in $G$, which yields two different simple paths $\omega_1,\omega_2$ in the Cayley graph starting at 1 and ending at $g\neq 1$. In this case, the elements $(\omega_1,g)$ and $(\omega_2,g)$ are $\sigma$-related, maximal and different, which proves $S$ is not F-inverse.

If $G$ does not have any non-trivial cycles (i.e., it is free), then for any $g\in G$ there is a unique geodesic $\omega$ going from 1 to $g$. In this case, the $\sigma$-class of elements of the form $(X,g)$ has a greatest element, namely $(\omega,g)$. □

As a particular example, let $G=\mathbb{Z}^2$. To fix notation, say $\mathbb{Z}^2$ is generated by the commuting elements $a,b$, and its unit is 1. Then $S$, i.e. the Meaking-Margolis extension of $\mathbb{Z}^2$, is easily seen to be generated (as an inverse monoid) by the elements

$$s=(\{1,a\},a)\quad\text{and}\quad t=(\{1,b\},b).$$

Indeed, observe first that we can form any element of the form $(X,1)\in S$ just by subsequent products between $s^is^{*i}$ and $t^it^{*i}$, where $i\in\mathbb{N}$. In order to produce $(X,g)$, we may then multiply $(X,1)$ by $s^it^j$ on the right, where $g=a^ib^j\in\mathbb{Z}^2$. This simple algorithm yields every possible element of $S$, proving that $S$ is, indeed, generated by $\{s,s^*,t,t^*\}$. As a visual help, Figure 5.7 shows how the multiplication in $S$ works in various simple cases.

The monoid $S$, however, is not good enough for our purposes. Indeed, observe that $(X,g)\ \mathcal{L}\ (Y,h)$ precisely if $g^{-1}X=h^{-1}Y$. Thus, since $X$ is a finite subgraph in $\mathbb{Z}^2$ containing 1 and $g$, there are only finitely many elements $\mathcal{L}$-related to $(X,g)$, thus proving that every Schützenberger graph of $S$ is finite. Hence any injective map, such as the quotient map $\sigma$, from a Schützenberger graph of $S$ into a metric space is a coarse embedding, for the Schützenberger distance is trivial (up to coarse equivalence).

In order to obtain an inverse semigroup $T$ that answers the questions at the beginning of the subsection negatively we shall produce an inverse semigroup whose Schützenberger graphs *lack edges*, in the sense that their vertices are significantly closer in the maximal homomorphic image $G(T)=\mathbb{Z}^2$ than they are in $T$. The precise example is given in the next proposition (see also Figure 5.8).

**Proposition 5.4.6.** *Let $S$ be the Meakin-Margolis expansion of $\mathbb{Z}^2$, and let $s,t$ be its canonical generators. Let $T$ be the quotient of $S$ by the relations $s^*s=ss^*=1$. Then:*

*(1) $T$ is a finitely generated inverse semigroup.*

*(2) The elements of $T$ can be seen as pairs $(X,g)$, where $g\in\mathbb{Z}^2$ and $X$ is a connected subgraph of $\mathbb{Z}^2$; containing 1 and $g$; with finitely many vertical edges; and containing the horizontal line through any of its vertices.*

*(3) $T$ is E-unitary, and its maximal homomorphic image is $\mathbb{Z}^2$.*

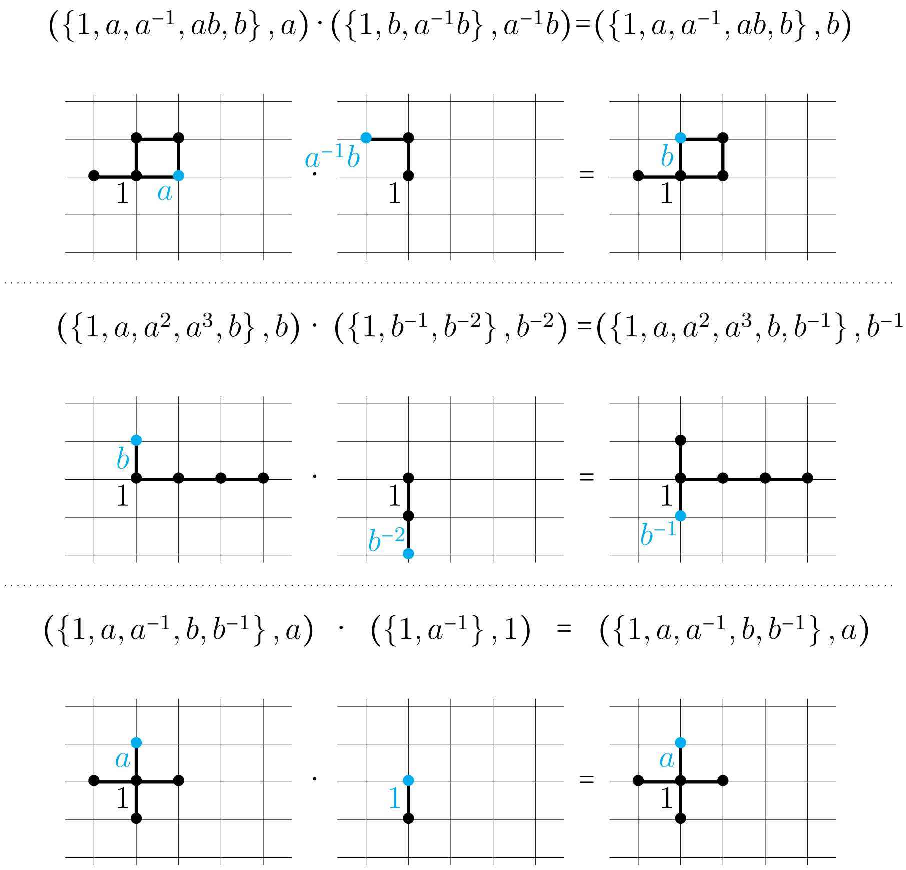


FIGURE 5.7: Graphical representation of some products of elements in the Meakin-Margolis expansion of $\mathbb{Z}^2 = \langle a,b \mid ab = ba\rangle$.

(4) *The quotient map $\sigma\colon T \to \mathbb{Z}^2$ does not coarsely embed the Schützenberger graph of $[t] \in T$ into $\mathbb{Z}^2$.*

*Proof.* (1) is clear by construction. For (2), let $\Phi(X)$ be the connected subgraph of $\mathbb{Z}^2$ formed by $X \subset \mathbb{Z}^2$ along with every horizontal line passing through any of its vertices. Then it can be readily seen that the map $T \ni [(X,g)] \mapsto (\Phi(X),g)$ is a bijective homomorphism. Note that the product on the right hand side is defined in the same way as the product in $S$.

Statement (3) follows from (2). Indeed, $T$ is E-unitary because the quotient map $\sigma\colon S \to \mathbb{Z}^2$ factors through $T$. Therefore the projection $\sigma$ is an idempotent-pure homomorphism.

Finally, (4) follows from a simple computation. Indeed, following the description in (2) observe

$$[t] = [(\{1,b\},b)] = (\mathbb{Z} \cup b\mathbb{Z}, b) \in T,$$

and hence

$$(X,g)\,\mathcal{L}\,[t] \;\Leftrightarrow\; g^{-1}X = b^{-1}(\mathbb{Z}\cup b\mathbb{Z}) \;\Leftrightarrow\; X = g\,(b\mathbb{Z}\cup\mathbb{Z}).$$

However, $X$ must contain both $g$ and 1, and hence $g$ itself must lie within $\mathbb{Z} \cup b^{-1}\mathbb{Z} \subset \mathbb{Z}^2$. It follows that the Schützenberger graph of the $\mathcal{L}$-class of $[t]$ is an *infinite rotated letter H* (see Figure 5.8). The projection $\sigma$ is then the *obvious one*, and it is manifestly not a coarse embedding, as noted in Figure 5.8. □

**Remark 5.4.7.** Observe that the monoid $T$ in Proposition 5.4.6 is not F-inverse. Indeed, following Proposition 5.2.12, if it was F-inverse then $\sigma$ would necessarily be a coarse embedding, contradicting (4) in Proposition 5.4.6. This, however, was already noted in Proposition 5.4.5.

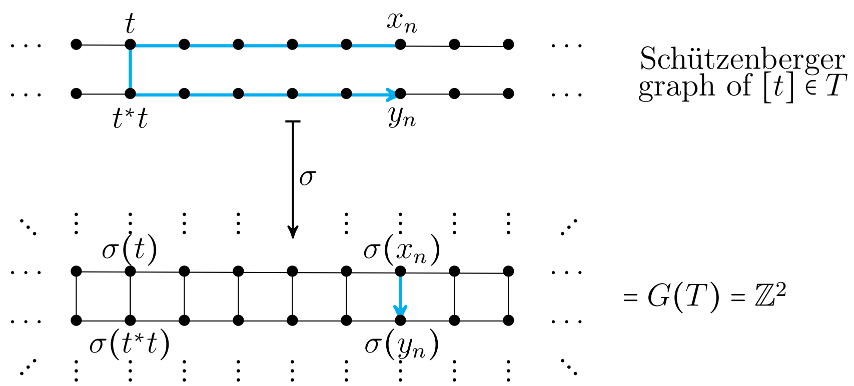


FIGURE 5.8: Representation of the quotient map $\sigma$ for a certain quotient $T$ of the Meakin-Margolis expansion of $\mathbb{Z}^2$, given in Proposition 5.4.6. The map is the obvious one from the picture, and the arrows represent geodesics in each graph. In this case, the points $x_n$ and $y_n$ are at distance $2n+1$, while $\sigma(x_n)$ and $\sigma(y_n)$ are constantly at distance 1, proving that $\sigma$ is not a coarse embedding.

### 5.4.5 Higson-Lafforgue-Skandalis' and Willett's example

This final class of examples reproduces the box space construction in the context of inverse semigroups. In particular, the main goal is to construct a locally finite inverse semigroup (and hence amenable, see Proposition 4.4.1) that does not have property A (in the sense that its Schützenberger graphs do not have uniform property A, see Definition 2.2.12). Note that this construction is, by now, well known, and has been replicated in several contexts. For instance, see [76, Example 4.4.7] (and [89, 53]) for the metric space construction; and see [3, Example 4.11]) (as well as [48, 115]) for the groupoid case.

Let $G$ be a finitely generated residually finite group, and let $\{N_i\}_{i\in\mathbb{N}}$ be a descending chain of normal subgroups of finite index of $G$ with trivial intersection. Consider then the semigroup $S_G := \sqcup_{i\in\mathbb{N}} G/N_i$, equipped with the operation

$$q_i(g) \cdot q_j(h) := q_{\min\{i,j\}}(gh),$$

where $q_i: G \to G/N_i$ is the canonical quotient map. Observe in particular that the $\mathcal{L}$ and $\mathcal{R}$ relations are equal, i.e., $S_G$ is a *bundle of groups*, also known as *Clifford semigroup*. This is of particular interest because of the following known reason (for a proof see [76, Theorem 4.4.6]):

**Proposition 5.4.8.** *Let $G$ be a residually finite group and $S_G$ be the Clifford semigroup constructed as above. Then $S_G$ has property A if and only if $G$ is amenable.*

Of particular interest is the case when $G = \mathbb{F}_2$ and $\{N_i\}_{i\in\mathbb{N}}$ is a descending chain of normal subgroups with trivial intersection. Then $S_{\mathbb{F}_2}$ is a locally finite Clifford semigroup without property A. Moreover, $C_r^*(S_{\mathbb{F}_2})$ is an AF-algebra, since $S_{\mathbb{F}_2}$ is locally finite.

Observe that $S_{\mathbb{F}_2}$, as constructed so far, does not meet the hypothesis of Theorem 5.3.10, since it does not admit a finite labeling. Indeed, any finite set of labels cover up to only finitely many subgroups $G/N_i$, and therefore cannot cover the whole semigroup.

In order to produce an inverse semigroup admitting a finite labeling consider $\overline{S_{\mathbb{F}_2}} := S_{\mathbb{F}_2} \sqcup \mathbb{F}_2$, where $q_i(g)h := q_i(gh)$ (and likewise for the right product). The semigroup $\overline{S_{\mathbb{F}_2}}$ is still Clifford, but is no longer locally finite (since $\mathbb{F}_2$ is not locally finite). In the same vein, observe the set of labels $\{a^{\pm 1}, b^{\pm 1}\} \subset \mathbb{F}_2$ label every edge in every Schützenberger graph of $\overline{S_{\mathbb{F}_2}}$, proving that $\overline{S_{\mathbb{F}_2}}$ admits a finite labeling. As such, $\overline{S_{\mathbb{F}_2}}$ meets the requirements of Theorem 5.3.10, and yields the following result.

**Proposition 5.4.9.** *The reduced C*-algebra of $\overline{S_{\mathbb{F}_2}}$ is not exact.*

In particular, the reduced C*-algebra of $\overline{S_{\mathbb{F}_2}}$ is non-nuclear. Moreover, Willett proved in [115] that, by choosing a particular descending chain of normal subgroups of $\mathbb{F}_2$, one may obtain a semigroup with the weak containment property. Indeed, see [115, Theorem 1.2] for a proof of the following, and relate it with the discussion in Section 4.5, particularly in Theorem 4.5.3.

**Theorem 5.4.10.** *If $\{N_i\}_{i\in\mathbb{N}}$ is as in Lemma 2.8 of [115], the reduced C*-algebra $C_r^*(\overline{S_{\mathbb{F}_2}})$ is non-nuclear, and coincides with the full C*-algebra $C^*(\overline{S_{\mathbb{F}_2}})$.*

## 5.5 Groupoid amenability, finite labelings and property A

This section is to Chapter 5 what Section 4.5 was to Chapter 4, that is, aims to relate the results proven so far with related results for groupoids. In particular, recall that Section 4.5 proved there is no relation between the amenability of the semigroup and the amenability of its universal groupoid. Actually, as stated in the end of Section 4.5, this is only natural, for one notion deals with traces and the other with nuclearity, and these are well known to be unrelated. However, in light of Theorem 5.3.10 the property A of $S$ should be related with the amenability of

$G_{\mathcal{U}}(S)$, for property A roughly means exactness while amenability of $G_{\mathcal{U}}(S)$ means nuclearity. This insight is, indeed, true, and the present section aims to prove it (see Theorem 5.5.4 below). Nevertheless, we first need some notation and some preliminary lemmas.

Recall (see Section 4.5) the construction of the universal groupoid $G_{\mathcal{U}}(S)$ of an inverse semigroup $S$. In particular, observe the idempotents of $S$ are included in the unit space of $G_{\mathcal{U}}(S)$ via

$$E \hookrightarrow \hat{E}_0 = G_{\mathcal{U}}(S)^{(0)}, \ \ e \mapsto e^{\uparrow} = \{f \in E \mid f \geq e\}.$$

and that $S$ acts on $\hat{E}_0$ via $\theta_s(\xi) := \{e \in E \mid e \geq sfs^* \text{ for some } f \in \xi\}$.

**Lemma 5.5.1.** *Let $s \in S$ and $e \in E$ be given. Then $e^{\uparrow} \in D^{\theta}_{s^*s}$ if, and only if, $s^*s \geq e$. Moreover*

$$\theta_s\left(e^{\uparrow}\right) = (ses^*)^{\uparrow} = \{f \in E \mid f \geq ses^*\}.$$

*Proof.* By definition $e^{\uparrow} \in D^{\theta}_{s^*s}$ if $s^*s \in e^{\uparrow}$, which happens precisely when $s^*s \geq e$. The other claim is shown similarly. □

**Lemma 5.5.2.** *For any $x \in S$ the map*

$$\left\{\left[s, (xx^*)^{\uparrow}\right] \in G_{\mathcal{U}}(S)\right\} \to \{z \in S \mid z^*z = x^*x\}, \ \ \textit{where } \left[s, (xx^*)^{\uparrow}\right] \mapsto sx$$

*is a bijection between $G_{\mathcal{U}}(S)_{(xx^*)^{\uparrow}}$ and the $\mathcal{L}$-class of $x$.*

*Proof.* First observe that the map is well defined, since $[s, (xx^*)^{\uparrow}] = [t, (xx^*)^{\uparrow}]$ if $sxx^* = txx^*$, which implies that $sx = tx$. In the same vein, $sx\mathcal{L}x$, for $s^*s \geq xx^*$ and, hence, $x^*s^*sx = x^*xx^*s^*sx = x^*x$. The map is also injective, since if $sx = tx$ then $sxx^* = txx^*$, which means that $[s, (xx^*)^{\uparrow}] = [t, (xx^*)^{\uparrow}]$ from the germ equivalence relation. Lastly, if $z \in S$ is $\mathcal{L}$-related with $x$ then the element $[zx^*, (xx^*)^{\uparrow}]$ maps to $zx^*x = z$, proving surjectivity. □

The characterization of the finite labeling condition given in Proposition 5.2.18 has a particular use in this context.

**Lemma 5.5.3.** *Let $S$ be a unital inverse semigroup. Then the following hold:*

*(1) Given $r \geq 0$, let $F_r$ be as in Proposition 5.2.18. Then the set*

$$K_r := \left\{\left[m, (s^*s)^{\uparrow}\right] \mid ms^*s = s, \ d(s^*s, s) \leq r \text{ and } m \in F_r\right\} \subset G_{\mathcal{U}}(S)$$

*is compact.*

*(2) If $K \subset G_{\mathcal{U}}(S)$ is compact then there is a finite $F \subset S$ such that if $[m, \xi] \in K$ then $m \in F$.*

*Proof.* For (1) let $\{[m_n, e_n^{\uparrow}]\}_{n\in\mathbb{N}} \subset K_r$ be a sequence. Since $F_r$ is finite we may suppose, without loss of generality, that $m_n = m$ for all $n \in \mathbb{N}$. Then observe that $\{[m, e_n^{\uparrow}]\}_{n\in\mathbb{N}}$ converges in $G_{\mathcal{U}}(S)$ if, and only if, $\{e_n^{\uparrow}\}_{n\in\mathbb{N}}$ converges in $\hat{E}_0$. Since $S$ is assumed unital the space $\hat{E}_0$ is compact, and whence the sequence $\{e_n^{\uparrow}\}_{n\in\mathbb{N}}$ has a convergent subsequence.

Now suppose (2) is false. Then for any $n \in \mathbb{N}$ there is $m_n \in S$ and $\xi_n \in \hat{E}_0$ such that $[m_n, \xi_n] \in K$ and $m_i \neq m_j$ for all $i \neq j$. This, however, contradicts the compactness of $K$, for the sequence $\{[m_n, \xi_n]\}_{n\in\mathbb{N}}$ has no convergent subsequence. □

**Theorem 5.5.4.** *Let $S = \langle K\rangle$ be a countable and discrete inverse semigroup, and let $G_{\mathcal{U}}(S)$ be its universal groupoid. Moreover, suppose that $(S, K)$ admits a finite labeling. If $G_{\mathcal{U}}(S)$ is amenable (as a groupoid), then $S$ has property A.*

*Proof.* First observe that, without loss of generality, we may suppose that $S$ is unital, since otherwise adding a unit to it does not change its property A. Now let $r, \varepsilon > 0$ be given. By Proposition 5.2.18 let $F_r$ be a finite set such that if $d(s^*s, s) \leq r$ then there is some $m \in F_r$ such that $ms^*s = s$. Following Lemma 5.5.3 the set

$$K_r := \left\{\left[m, (s^*s)^{\uparrow}\right] \mid ms^*s = s,\ d(s^*s, s) \leq r \text{ and } m \in F_r\right\} \subset G_{\mathcal{U}}(S)$$

is compact. Therefore there is a non-negative $\eta\colon G_{\mathcal{U}}(S) \to [0,1]$ witnessing the $(K_r, \varepsilon)$-amenability of $G_{\mathcal{U}}(S)$. Consider the function

$$\xi\colon S \to \ell^1(S), \text{ where } \xi_x(z) := \begin{cases} \eta\left(\left[zx^*, (xx^*)^{\uparrow}\right]\right) & \text{if } z^*z = x^*x, \\ 0 & \text{otherwise.}\end{cases}$$

We claim that $\xi$ witnesses the property A of $S$ with respect to $r$ and $\varepsilon$. Indeed, first note that $\xi_x$ is a non-negative function of norm almost 1, for if $L_{x^*x}$ denotes the $\mathcal{L}$-class of $x$ then

$$\|\xi_x\|_1 = \sum_{z\in L_{x^*x}} \xi_x(z) = \sum_{z\in L_{x^*x}} \eta\left(\left[zx^*, (xx^*)^{\uparrow}\right]\right) = \sum_{\left[s,(xx^*)^{\uparrow}\right]\in G_{\mathcal{U}}(S)} \eta\left(\left[s, (xx^*)^{\uparrow}\right]\right) \approx_{\varepsilon} 1,$$

where the third equality follows from the bijection in Lemma 5.5.2. Likewise, since $\eta$ is compactly supported, by Lemma 5.5.3 there is some finite set $F_\eta$ such that if $\xi_x(z) \neq 0$ then $zx^* \in F_\eta$. Then

$$d(x, z) = d(xx^*, zx^*) \leq \max\{d(m^*m, m) \mid m \in F_\eta\} =: c < \infty,$$

which proves that the support of $\xi_x$ is contained in the ball of radius $c$ around $x$ (note that $c$ is independent of $x$).

Lastly, let $x, y \in S$ be at distance $r$. Then $r \geq d(x, y) = d(xx^*, yx^*)$ and therefore, by construction, there is some $m \in F_r$ such that $mxx^* = yx^*$. In particular $y = mx$

and, hence,

$$\begin{aligned}\|\xi_x - \xi_y\|_1 &= \sum_{z\in L_{x^*x}} \left|\eta\left(\left[zx^*, (xx^*)^\uparrow\right]\right) - \eta\left(\left[zy^*, (yy^*)^\uparrow\right]\right)\right| \\ &= \sum_{z\in L_{x^*x}} \left|\eta\left(\left[zx^*, (xx^*)^\uparrow\right]\right) - \eta\left(\left[zx^*, (xx^*)^\uparrow\right]\cdot\left[m, (xx^*)^\uparrow\right]^{-1}\right)\right| \leq \varepsilon\end{aligned}$$

since $[m, (xx^*)^\uparrow] \in K_r$. □

Note that, in the proof above, we only have $\|\xi_x\|_1 \approx 1$. It is, however, well known (and routine to prove) that this is enough for property A. We end the chapter with the following remarks, all of which state that Theorem 5.5.4 is not surprising, and its reciprocal does not hold.

**Remark 5.5.5.** Observe that there are inverse semigroups with property A whose associated universal groupoid is not amenable. Indeed, the free group $\mathbb{F}_2$ has property A, since it is a tree, but it is not amenable.

**Remark 5.5.6.** The proof of Theorem 5.5.4 also follows from the work done in Section 5.3.2. Indeed, it is well known (see [81, Theorem 4.4.2]) that $C_r^*(S) = C_r^*(G_{\mathcal{U}}(S))$. By Theorem 5.3.7 we observe that $C_r^*(S)$ is exact whenever $S$ has property A, and it is well known that $C_r^*(G_{\mathcal{U}}(S))$ is nuclear if, and only if, $G_{\mathcal{U}}(S)$ is amenable (see Theorem 4.5.2). Therefore, Theorem 5.5.4 above, in C*-language, is saying that *nuclearity implies exactness*. From this point of view it is then again obvious that the reciprocal of Theorem 5.5.4 cannot hold, since there are non-nuclear exact algebras, such as $C_r^*(\mathbb{F}_2)$.

**Remark 5.5.7.** On a technical level, it is also clear why the reciprocal of Theorem 5.5.4 cannot hold. Indeed, following the proof one sees that the property A of $S$ is assured by the amenability of $G_{\mathcal{U}}(S)$ but only on *some* filters, namely those of the form $e^\uparrow$ for $e \in E$. Meanwhile, for $G_{\mathcal{U}}(S)$ to be amenable one needs $\eta$ to be defined on *all* filters and in a continuous manner. Therefore it is very much not clear how one could go from property A of $S$ to amenability of $G_{\mathcal{U}}(S)$ without resorting to some internal structure of the semigroup (such as having finitely many projections).

# Chapter 6
# Quasi-diagonality and norm approximations

In the appendix to [46] Rosenberg showed that any countable discrete group $G$ with a quasi-diagonal left regular representation is amenable. In particular, this result implies that $G$ is amenable whenever $C_r^*(G)$ is a quasi-diagonal C*-algebra (cf., [20, Corollary 7.1.17]). The reverse implication, boldly conjectured by Rosenberg to hold, was famously shown in [107, Corollary C]. In particular, and going back to the 2-norm approximations of $C_r^*(S)$ given in Chapter 4, one may wonder whether there is any relation between the amenability of an inverse semigroup and the quasi-diagonality of its reduced C*-algebra. This final chapter addresses this question.

First recall that the quasi-diagonality of a unital and separable C*-algebra can be defined in terms of a sequence of unital completely positive (u.c.p.) maps $\varphi_n\colon A \to M_{k_n}$ that are asymptotically multiplicative and asymptotically isometric in operator norm (see Definition 2.3.10). Moreover, recall that the *reduced C*-algebra of an inverse semigroup* is the C*-algebra generated by the left regular representation (see Proposition 3.3.5):

$$C_r^*(S) := C^*\left(\{V_s\}_{s\in S}\right) \subset \mathcal{B}\left(\ell^2 S\right).$$

This chapter is organized as follows. In Section 6.1 we introduce a class of inverse semigroups that yield AF-algebras, hence providing a first class of semigroups with quasi-diagonal reduced C*-algebras. Then Section 6.2 gives a partial answer to Rosenberg's conjecture in the context of inverse semigroups by proving that domain-measurability is a necessary condition for the quasi-diagonality of $C_r^*(S)$ (see Theorem 6.2.1). Lastly, Section 6.3 addresses the reciprocal of Theorem 6.2.1, giving some sufficient conditions that guarantee the quasi-diagonality of $C_r^*(S)$.

## 6.1 AF monoids

The present section is rather laid back and expository, and we include it for the sake of the reader. Recall that a C*-algebra $A$ is said to be *approximately finite dimensional*, or *AF* for short, if there is an increasing sequence of finite dimensional sub-algebras $A_n \subset A$ such that $A$ is the norm-closure of $\cup_{n\in\mathbb{N}} A_n$. Examples of AF algebras include the finite-dimensional algebras, the continuous functions on the Cantor set, or the algebra $\mathcal{Q} := \otimes_{n\in\mathbb{N}} M_n$, usually called the *universal UHF algebra.*

Following the idea that AF structures are inductive limits of finite ones, Lawson and Scott introduced in [59, Section 3] what they termed *AF-monoids* as those monoids that can be constructed as inductive limits of finite monoids. For that, denote by $I_p$ the inverse semigroup of partial bijections in the finite set $\{1,\dots,p\}$. Then we say $S$ is an *AF monoid* if $S$ can be written as $\cup_{n\in\mathbb{N}} S_n$, where $S_n$ is a finite monoid of the form

$$S_n = I^{(n)}_{m_1} \times \cdots \times I^{(n)}_{m_n}.$$

One important feature of these monoids, and one very much exploited in [59], is the fact that this construction retains a lot of the ideas behind AF algebras. For instance, there is a homomorphism $I_n \to I_m$ only if $n$ divides $m$ (see, e.g., [59, Lemma 3.2]). This fact allows to classify AF monoids by their *Bratteli diagrams*, just as one would do for AF-algebras. Moreover, this class of monoids represent the first class of examples of *quasi-diagonal semigroups*, that is, semigroups with quasi-diagonal reduced C*-algebras.

**Proposition 6.1.1.** *The reduced C*-algebra of an AF-monoid is AF, and hence quasi-diagonal. In general, the reduced C*-algebra of a locally finite inverse semigroup is also quasi-diagonal.*

*Proof.* The first statement follows from the observation that if $S = \cup_{n\in\mathbb{N}} S_n$ is an AF-monoid then $C^*_r(S_n)$ are finite-dimensional C*-algebras. Therefore $C^*_r(S)$ is the norm-closure of $\cup_{n\in\mathbb{N}} C^*_r(S_n)$, proving that $C^*_r(S)$ is indeed AF. For the second statement, observe that if $S$ is a locally finite semigroup (cf. Section 4.4) then $C^*_r(S)$ always has an approximate unit of central projections, i.e., a sequence of projections converging strongly to the identity and commuting with $C^*_r(S)$. □

## 6.2 Rosenberg's conjecture for inverse semigroups

This last-before-last section of the thesis addresses Rosenberg's conjecture (see [46] and [107, Corollary C]) in the case of discrete and countable inverse semigroups. In particular, note that the following theorem can only be true for the reduced C*-algebras (either in this context or for groups), since the uniform Roe algebra $\ell^\infty(S) \rtimes_r S$ are almost never finite (see [95]), and thus almost never quasi-diagonal (see [20, Proposition 7.1.15]).

**Theorem 6.2.1.** *Let $S$ be a countable and discrete inverse monoid. Then:*

*(1) If $C^*_r(S)$ is quasi-diagonal then $S$ is domain measurable.*

*(2) If $C^*_r(S)$ is quasi-diagonal and $S$ has a minimal projection then it is amenable.*

*(3) There are amenable inverse semigroups with and without minimal projections with non-quasi-diagonal reduced C*-algebras.*

*Proof.* As is customary in the literature, we will denote by $\mathrm{tr}_{k_n}$ the normalized trace in the matrix algebra $M_{k_n}$, while Tr will stand for the usual trace, i.e., $\mathrm{tr}_{k_n}(\cdot) = \mathrm{Tr}(\cdot)/k_n$ (cf. Section 2.3).

The proof of (1) follows the same strategy as Proposition 7.1.16 in [20]. Indeed, let $\varphi_n : C_r^*(S) \to M_{k_n}$ be a sequence of u.c.p. maps that are asymptotically multiplicative and isometric. Then any cluster point $\tau$ of $\{\mathrm{tr}_{k_n} \circ \varphi_n\}_{n\in\mathbb{N}}$ is an amenable trace of $C_r^*(S)$. Thus let $\Phi$ be any hypertrace extending $\tau$, and consider the measure $\mu(A) := \Phi(p_A)$. Then

$$\mu(B) = \psi(V_{s^*s} p_B) = \psi(V_s p_B V_{s^*}) = \psi(p_{sB})$$

for every $B \subset D_{s^*s}$, proving that $\mu$ indeed is a domain measure of $S$.

For (2), let $e_0 \in E(S)$ be the minimal projection and consider $\varphi_n : C_r^*(S) \to M_{k_n}$ a sequence of unital completely positive (u.c.p.) maps that are asymptotically multiplicative and asymptotically isometric. To prove (2) we will construct a new sequence of u.c.p. maps $\phi_n$ that are asymptotically multiplicative, asymptotically isometric and with asymptotically normalized trace at $V_{e_0}$. For this, recall that, by Lemma 4.2.19, $e_0$ commutes with every $s \in S$.

Observe that $\varphi_n(V_{e_0})$ is a positive matrix, whose norm is greater than $1-\varepsilon_n$ and such that $\|\varphi_n(V_{e_0}) - \varphi_n(V_{e_0})\varphi_n(V_{e_0})\| < \varepsilon_n$ for some $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ when $n \to \infty$. Assuming that $\varepsilon_n < 1/4$ a routine exercise shows that the spectrum of $\varphi_n(V_{e_0})$ is contained in $[0, \delta_n) \cup (1-\delta_n, 1]$ where $\delta_n = \frac{1}{2}(1 - \sqrt{1-4\varepsilon_n})$. Let $r_n$ be the number of eigenvalues of $\varphi_n(V_{e_0})$ in $(1-\delta_n, 1]$, and note that $r_n \geq 1$ for large $n \in \mathbb{N}$, since $\|\varphi_n(V_{e_0})\| \geq 1 - \varepsilon_n$. Let $W_n \subset \mathbb{C}^{k_n}$ be the subspace generated by the eigenvectors of $\varphi_n(V_{e_0})$ of eigenvalues close to 1. Finally, let $q_n : \mathbb{C}^{k_n} \to \mathbb{C}^{r_n}$ be a linear map such that $q_n|_{W_n}$ is an isometry onto $\mathbb{C}^{r_n}$ and $q_n|_{W_n^\perp} = 0$, i.e.,

$$q_n : \mathbb{C}^{k_n} \to \mathbb{C}^{r_n}, \quad q_n q_n^* = 1_{r_n} \text{ and } q_n^* q_n = p_{W_n}.$$

For each $n \in \mathbb{N}$ let $m_n \in \mathbb{N}$ be large enough so that

$$\frac{m_n r_n}{k_n + m_n r_n}(1 - \delta_n) \geq 1 - 2\,\delta_n. \tag{6.1}$$

Consider the maps:

$$\phi_n : C_r^*(S) \to M_{k_n + m_n r_n}, \quad a \mapsto \varphi_n(a) \oplus \left(q_n \varphi_n(a) q_n^* \otimes 1_{m_n}\right). \tag{6.2}$$

By construction it is clear that the maps $\phi_n$ are unital, completely positive and asymptotically isometric. They are also asymptotically multiplicative. For this first observe that, using the minimality of $e_0$ and Lemma 4.2.19, $V_{e_0}$ commutes with every $a \in C_r^*(S)$. Thus, by summing and substracting $\varphi_n(V_{e_0})\varphi_n(a)$, $\varphi_n(a)\varphi_n(V_{e_0})$ and

$\varphi_n(V_{e_0}a)$, we have

$$\begin{aligned}\|q_n^* q_n \varphi_n(a) - \varphi_n(a) q_n^* q_n\| &\leq 2\|\varphi_n(a)\| \, \|\varphi_n(V_{e_0}) - q_n^* q_n\| \\ &\quad + \|\varphi_n(V_{e_0})\varphi_n(a) - \varphi_n(V_{e_0}a)\| \\ &\quad + \|\varphi_n(a)\varphi_n(V_{e_0}) - \varphi_n(aV_{e_0})\| \\ &\leq 2\|a\| \, \|\varphi_n(V_{e_0}) - q_n^* q_n\| \\ &\quad + \|\varphi_n(V_{e_0})\varphi_n(a) - \varphi_n(V_{e_0}a)\| \\ &\quad + \|\varphi_n(a)\varphi_n(V_{e_0}) - \varphi_n(aV_{e_0})\| \xrightarrow{n\to\infty} 0.\end{aligned}$$

This asymptotic commutation gives the asymptotic multiplicativity of $\phi_n$ by a straightforward computation, since for any $a, b \in C_r^*(S)$

$$\begin{aligned}\|\phi_n(ab) - \phi_n(a)\phi_n(b)\| &\leq \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \\ &\quad + \|q_n\varphi_n(ab)q_n^* - q_n\varphi_n(a)q_n^* q_n\varphi_n(b)q_n^*\| \\ &\leq 2\,\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \\ &\quad + \|b\| \, \|q_n^* q_n\varphi_n(a) - \varphi_n(a)q_n^* q_n\| \xrightarrow{n\to\infty} 0.\end{aligned}$$

Furthermore, the maps $\phi_n$ have the desired asymptotically normalized trace property at $V_{e_0}$:

$$\begin{aligned}1 \geq \mathrm{tr}(\phi_n(V_{e_0})) &= \mathrm{tr}(\varphi_n(V_{e_0})) + m_n \, \mathrm{tr}(q_n\varphi_n(V_{e_0})q_n^*) \\ &= \frac{m_n}{k_n + m_n r_n}\mathrm{Tr}(q_n^* q_n \varphi_n(V_{e_0})) \\ &\geq \frac{m_n r_n}{k_n + m_n r_n}(1 - \delta_n) \geq 1 - 2\,\delta_n \xrightarrow{n\to\infty} 1,\end{aligned}$$

where the last inequality follows from the choice of $m_n$, see Eq. (6.1). By the discussion in (1) and Theorem 4.3.14, any cluster point $\Phi$ of $\{\mathrm{tr}_{k_n+m_n r_n} \circ \phi_n\}_{n\in\mathbb{N}}$ will give an amenable trace of $C_r^*(S)$ normalized at $V_{e_0}$. Indeed, observe that

$$\mu(D_{e_0}) = \Phi(V_{e_0}) = \lim_{n\to\infty} \mathrm{tr}_{k_n+m_n r_n}(\phi_n(V_{e_0})) = 1.$$

We conclude that $\mu$ is a domain-measure localized at $D_{e_0}$ and, by Proposition 4.2.20 and Theorem 4.3.14, $S$ is then amenable.

Finally, for (3), consider the bicyclic inverse monoid $S = \langle a, a^* \mid a^*a = 1\rangle$. It is routine to show that $E(S) = \{1, aa^*, a^2(a^*)^2, \dots\}$ and, thus, $S$ has no minimal projection. Moreover, it is amenable (see [31, pp. 311]) and $C_r^*(S)$ is not quasidiagonal (since it is not even finite, see [20, Proposition 7.1.15]).

For the case when $S$ does have a minimal projection, consider the semigroup $T = \mathbb{F}_2 \sqcup \{0\}$, where 0 is a zero element. Since $T$ has a zero element it follows that $T$ is amenable (take $\mu$ atomic with total mass at 0) and has a minimal projection (namely 0). Furthermore, note that $C_r^*(S)$ is not quasidiagonal, since it contains the reduced group C*-algebra of $\mathbb{F}_2$, which is not quasidiagonal (see [20, Proposition 7.1.10] and [107, Corollary C]). □

**Remark 6.2.2.** The condition of the existence of a minimal projection $e_0$ in Theorem 6.2.1 (2) is assumed to guarantee that it commutes with every element $s \in S$. In particular, the proof of (2) in Theorem 6.2.1 can also be applied to the more general setting of a unital and separable C*-algebra $A$ and a central projection $p \in A$. Indeed, suppose that $A$ is also quasi-diagonal. Then, by the same argument, the construction of Eq. (6.2) gives a quasi-diagonal approximation of $A$ with asymptotic normalized trace at $p$. In both cases, however, the condition that $ap = pa$ for every $a \in A$ is essential.

## 6.3 Idempotents, $\mathcal{D}$-classes and quasi-diagonality

This last section of the chapter is qualitatively different from the rest. Indeed, all the contents from this section are part of an ongoing line of research, and hence remain, as of time of writing, unpublished. It thus might seem we have no clear path in mind and only meander around, trying to reach a goal. As such, and roughly speaking, this section might be closer to a discussion or a lecture than to an actual paper or thesis. In particular, we now try to address the reciprocal of Theorem 6.2.1, that is:

*Give algebraic conditions on a (countable and discrete) inverse semigroup $S$ that guarantee that $C_r^*(S)$ is a quasi-diagonal C*-algebra.*

There are (at the time of writing) two main strategies to prove that a given C*-algebra $A$ is quasi-diagonal: the *direct* strategy and the *TWW* strategy (this nomenclature is not customary in the literature).

(1) If the inner structure of $A$ is known and is *nice* one may explicitly construct quasi-diagonal approximations of the C*-algebra. For instance, any algebra containing a central approximate unit of finite dimensional projections is quasi-diagonal, as is any C*-algebra with a separating sequence of finite dimensional quotients (these are called *residually finite dimensional*, see [20, Proposition 7.1.7]).

(2) If the inner structure of $A$ is unknown, or if it is not *obviously* quasi-diagonal, then one may try to apply the recent Tikuisis-White-Winter theorem (see Theorem 2.3.11). For instance, this is the only known way to prove that an amenable discrete group has a quasi-diagonal C*-algebra.

Therefore, since we wish to study the quasi-diagonality of reduced inverse semigroup C*-algebras we have to resort to strategy (2) above, as the inner structure of the algebra is mostly unknown (even modulo the quasi-diagonality of group C*-algebras). In fact, using the main theorem of [107], Rainone and Sims study in [87] the quasi-diagonality of reduced groupoid C*-algebras. In particular, in [87, Theorem 6.5] the *minimality* assumption on the groupoid is fundamental, coming into play in order to prove the faithfulness assumption of the constructed trace.

In our context we could try to use Theorem 6.5 in [87] by means of Paterson's universal groupoid $G_{\mathcal{U}}(S)$ (cf. Section 4.5). As Proposition 6.3.1 shows this is not a good idea, as this method imposes a very restrictive set of conditions on the inverse semigroup $S$ (see also Proposition 6.3.4 and Figure 6.1).

Recall that two idempotents $e, f \in E(S)$ are $\mathcal{D}$-related if, and only if, there is some $s \in S$ such that $e = s^*s$ and $f = ss^*$. This relation can be extended to the whole $S$ by stating that $\mathcal{D}$ contains $\mathcal{L}$, meaning every idempotent $e \in E(S)$ is $\mathcal{D}$-related with everything in its $\mathcal{L}$-class. Moreover, we say an inverse semigroup $S$ is *bisimple* if it has exactly one $\mathcal{D}$-class (see, for instance, [58, p. 85]). Likewise, $S$ is *0-bisimple* if it is either bisimple or has a zero element and exactly two $\mathcal{D}$-classes. Note that in the latter case one of the $\mathcal{D}$-classes is necessarily formed by the zero element alone. Lastly, we say a groupoid $G$ is *minimal* if the range of $G_u$ is dense in $G^{(0)}$ for every unit $u \in G^{(0)}$, where $G_u$ denotes the set of elements whose source is $u$.

**Proposition 6.3.1.** *Let $S$ be a countable and discrete inverse semigroup. Suppose that $G_{\mathcal{U}}(S)$ is minimal (as a groupoid) and $C_r^*(S)$ is stably finite. Then the following assertions hold:*

*(a) The partial order in $S$ restricts to equality in the $\mathcal{D}$-classes of $S$, that is, if $s \geq t$ and $s\mathcal{D}t$ then $s = t$ for every $s, t \in S$.*

*(b) $S$ is 0-bisimple.*

*In particular, if $S$ is not a group then it must contain a zero element.*

*Proof.* Assertion (a) follows from $C_r^*(S)$ being stably finite. In fact, let $e, f \in E(S)$ be two idempotents such that $e \geq f$ and $e\mathcal{D}f$. Then $e = f$, as otherwise this would contradict the stable finiteness of $C_r^*(S)$. In general, given any pair $s, t \in S$ such that $s \geq t$ it follows that $s^*s \geq t^*t$. If, moreover, $s\mathcal{D}t$ then $s^*s\mathcal{D}t^*t$ and, hence, $s^*s = t^*t$, which implies that $s = ss^*s = st^*t = t$.

The proof of condition (b) follows from (a) and the minimality of $G_{\mathcal{U}}(S)$. Indeed, note that, in order to prove that $S$ is 0-bisimple it suffices to show that any two non-zero idempotents $e, f \in E(S) \setminus \{0\}$ are $\mathcal{D}$-related. As $G_{\mathcal{U}}(S)$ is minimal there are elements $\{s_n\}_{n\in\mathbb{N}}, \{t_k\}_{k\in\mathbb{N}} \subset S$ such that

$$e^{\uparrow} \in D^{\theta}_{s_n^*s_n} \text{ (i.e. } s_n^*s_n \geq e) \text{ and } r\left(\left[s_n, e^{\uparrow}\right]\right) = (s_n e s_n^*)^{\uparrow} \xrightarrow{n\to\infty} f^{\uparrow},$$
$$f^{\uparrow} \in D^{\theta}_{t_k^*t_k} \text{ (i.e. } t_k^*t_k \geq f) \text{ and } r\left(\left[t_k, f^{\uparrow}\right]\right) = (t_k f t_k^*)^{\uparrow} \xrightarrow{k\to\infty} e^{\uparrow}.$$

In particular note the latter convergences imply that $t_k^*t_k \geq f \geq s_n e s_n^*$ for large $n, k \in \mathbb{N}$, and whence $(s_n e s_n^*)^{\uparrow} \in D^{\theta}_{t_k^*t_k}$. Therefore

$$e\,\mathcal{D}\,s_n e s_n^*\,\mathcal{D}\,t_k s_n e s_n^* t_k^* \quad \text{and} \quad e \geq t_k f t_k^* \geq t_k s_n e s_n^* t_k^*.$$

From item (a) it follows that $e = t_k f t_k^* = t_k s_n e s_n^* t_k^*$ which, in particular, proves that $e = (t_k f)(t_k f)^*$ and $f = (t_k f)^*(t_k f)$, as desired.

Lastly, observe that if $S$ is not a group then it must contain two different projections $e, f \in E(S)$. If neither them nor their product is equal to zero then $e \geq ef$ and $e\mathcal{D}ef$ which, by (a), implies that $e = ef = f$, yielding a contradiction. $\square$

**Remark 6.3.2.** In Proposition 6.3.1 above we would have liked to prove that if $C_r^*(S)$ is furthermore assumed to be quasi-diagonal then every subgroup of $S$ is amenable. Observe this trivially holds if $S$ is a group, as then it must amenable, and amenability is preserved under taking subgroups.

Proposition 6.3.1 shows that the $TWW$-strategy, see (2) at the beginning of the section, does not cover the intended results in the context of inverse semigroups. Indeed, the class of inverse semigroups that are as in Proposition 6.3.1 is rather small, as Proposition 6.3.4 and Figure 6.1 below show.

**Lemma 6.3.3.** *Let $S$ be an inverse semigroup, and let $D \subset S$ be a $\mathcal{D}$-class. Then any two $\mathcal{H}$-classes in $D$ are in bijective correspondence. Moreover, in case both classes contain an idempotent then the bijection may be taken to be a group isomorphism.*

*Proof.* The proof is fairly similar to that of Proposition 5.1.9. Fix an idempotent $e_0 \in D$ and, given any projection $f \in D$, let $r_f \in D$ be such that $r_f^* r_f = e_0$ and $r_f r_f^* = f$. We may further assume that $r_{e_0} = e_0$. Then, letting $H_{e_0}$ be the $\mathcal{H}$-class of $e_0$, the map

$$\phi \colon D \to H_{e_0}, \ \text{ where } \ s \mapsto r_{ss^*}^* s r_{s^*s}$$

restricts to the desired bijections in every $\mathcal{H}$-class $H \subset D$. Indeed, first note that

$$\begin{aligned}\phi(s)^* \phi(s) = (r_{ss^*}^* s r_{s^*s})^* (r_{ss^*}^* s r_{s^*s}) &= r_{s^*s}^* s^* r_{ss^*} r_{ss^*}^* s r_{s^*s} \\ &= r_{s^*s}^* s^* s s^* s r_{s^*s} = r_{s^*s}^* r_{s^*s} r_{s^*s}^* r_{s^*s} = e_0,\end{aligned}$$

and that, by a similar computation, $\phi(s)\phi(s)^* = e_0$. Likewise, if $s, t \in D$ are $\mathcal{H}$-related and $\phi(s) = \phi(t)$ then

$$s = (ss^*)s(s^*s) = r_{ss^*}(r_{ss^*}^* s r_{s^*s}) r_{s^*s}^* = r_{tt^*}(r_{tt^*}^* t r_{t^*t}) r_{t^*t}^* = tt^*tt^*t = t,$$

which proves that $\phi$ is injective on every $\mathcal{H}$-class of $D$. Moreover, given any $h \in H_{e_0}$ note that $\phi(r_f h r_e^*) = h$, where $(f, e)$ is the ordered pair of projections that determines $H$. Lastly, in case $e = f$ then $H$ contains an idempotent, namely $e$ itself, and hence is a (maximal) subgroup of $S$. In this case

$$\phi(gh) = r_e^* g h r_e = r_e^* g e h r_e = (r_e^* g r_e)(r_e^* h r_e) = \phi(g)\phi(h)$$

for every $g, h \in H$, proving that $\phi$ indeed restricts to a group isomorphism. $\square$

**Proposition 6.3.4.** *Let $S$ be an inverse semigroup, and let $E^+ := E(S) \setminus \{0\}$ be the set of non-zero projections. Suppose $S$ is 0-bisimple and the partial order of $S$ reduces to equality in the $\mathcal{D}$-classes of $S$. Then either $S$ is a group or it is isomorphic to*

$$E^+ \times H \times E^+ \sqcup \{0\} = \{(f, h, e) \mid e, f \in E^+ \text{ and } h \in H\} \sqcup \{0\},$$

*where $H$ is a subgroup of $S$, and the operation in $E^+ \times H \times E^+$ is defined by*

$$(f_2, h_2, e_2)(f_1, h_1, e_1) := \begin{cases} (f_2, h_2 h_1, e_1) & \text{if } e_2 = f_1 \\ 0 & \text{otherwise}, \end{cases}$$

*Proof.* The isomorphism is given by a *coordinatized* version of the map in the proof of Lemma 6.3.3. Indeed, consider

$$\Phi\colon S \to E^+ \times H \times E^+ \sqcup \{0\}, \quad \text{where } \Phi(s) = \begin{cases} (ss^*, \phi(s), s^*s) & \text{if } s \neq 0, \\ 0 & \text{if } s = 0, \end{cases}$$

where $\phi$ is as in the aforementioned proof and $H$ is the $\mathcal{H}$-class of any fixed non-zero idempotent $e_0 \in E^+$. In particular, note $\Phi$ is an homomorphism since for any $s, t \in S$ we have that $st = 0$ unless $tt^* = s^*s$, and therefore

$$\begin{aligned} \Phi(st) = (ss^*, \phi(st), t^*t) &= (ss^*, \phi(s)\phi(t), t^*t) \\ &= (ss^*, \phi(s), s^*s)(tt^*, \phi(t), t^*t) = \Phi(s)\Phi(t). \end{aligned}$$

The fact that $\Phi$ is bijective can be proven in a similar fashion to the proof of Lemma 6.3.3. ☐

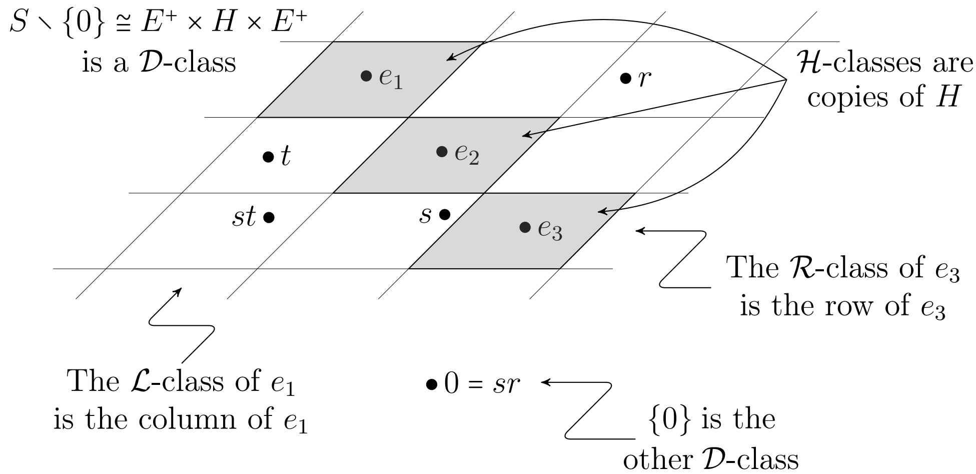


Figure 6.1: Visual representation of a semigroup $S$ as in Proposition 6.3.4 that is not a group. Particularly, $S$ necessarily has a zero element. In this case, note $sr = 0$ as $s^*s = e_2$, $rr^* = e_1$ and $e_1e_2 = 0$.

We now change directions, and proceed onto the proof of the main theorem of the section, namely Theorem 6.3.8, which follows the *direct* strategy and achieves a partial result about the quasi-diagonality of inverse semigroup C*-algebras.

The following two lemmas are rather basic, but we include them here for completeness and future reference.

**Lemma 6.3.5.** *Let $S$ be an inverse semigroup, $s \in S$ be an element and $H \subset S$ be an $\mathcal{H}$-class. Then the following hold:*

*(1) $H \subset D_{s^*s}$ if, and only if, $H \cap D_{s^*s} \neq \varnothing$.*

*(2) If $H \subset D_{s^*s}$ then $sH$ is an $\mathcal{H}$-class within the same $\mathcal{L}$-class as $H$.*

*Proof.* (1) follows from the observation that $h_1 h_1^* = h_2 h_2^*$ for every $h_1, h_2 \in H$. Thus if $h_1 \in H \cap D_{s^*s}$ then $h_2 h_2^* = h_1 h_1^* \leq s^*s$, proving that $h_2 \in D_{s^*s}$ for any $h_2 \in H$.

For (2), take any pair of elements $sh_1, sh_2 \in sH$. Then $(sh_1)^*(sh_1) = h_1^* s^* s h_1 = h_1^* h_1$ and, by the same computation, $(sh_2)^*(sh_2) = h_2^* h_2$. Likewise, $(sh_1)(sh_1)^* = sh_1 h_1^* s^* = sh_2 h_2^* s^*$. Therefore $sH \subset H'$ for some $\mathcal{H}$-class $H'$ and, moreover, observe that $sH = H'$, as left multiplication by $s^*$ defines the inverse map (from $H'$ to $H$). Lastly, note that $H$ and $sH$ belong to the same $\mathcal{L}$-class by Lemma 4.3.3. □

**Lemma 6.3.6.** *Let $a, v \in \mathcal{B}(\mathcal{H})$ be bounded operators on a Hilbert space. Suppose that $v$ is a partial isometry and $a(1 - vv^*) = 0$. Then $\|a\| = \|av\|$.*

*Proof.* Note that $a = avv^* + a(1 - vv^*) = avv^*$, and therefore $\|a\| = \|avv^*\| \leq \|av\|$. The other inequality is obvious. □

The upcoming lemma is the main technical result of the section, and the main tool to prove Theorem 6.3.8 below. Its proof, though rather straightforward, is long, messy and requires some notation. Recall that the *right regular representation* $W$ of $S$ is defined analogously as the left regular representation:

$$W_s \colon \ell^2 S \to \ell^2 S, \quad \delta_x \mapsto \begin{cases} \delta_{xs^*} & \text{if } xs^*s = x, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

**Lemma 6.3.7.** *Let $S$ be a countable inverse semigroup, and let $D \subset S$ be a $\mathcal{D}$-class with finitely many projections. Fix an idempotent $e_0 \in D$, and suppose its $\mathcal{H}$-class is amenable (as a group). Then there is a sequence of finite dimensional orthogonal projections $\{q_n\}_{n \in \mathbb{N}} \subset \mathcal{B}(\ell^2 S)$ satisfying the following conditions:*

*(1) $q_n \cdot 1_D = q_n = 1_D \cdot q_n$, where $1_D$ denotes the characteristic function of $D$.*

*(2) $q_n$ converges strongly to $1_D$.*

*(3) $\|V_s q_n - q_n V_s\| \to 0$ as $n \to \infty$ for every $V_s \in C_r^*(S)$.*

*Proof.* Let $E(D) = E(S) \cap D$ be the set of idempotents within $D$, which is finite by assumption. For each $e \in E(D)$ denote by $L_e$ and $R^e$ respectively the $\mathcal{L}$-class and $\mathcal{R}$-class of $e$. Likewise, let $H_e^f = L_e \cap R^f$ be the different $\mathcal{H}$-classes in $D$, and in particular note that $e \in H_e^e$. As in the proof of Lemma 6.3.3, let $r_{e_0} := e_0$, and for each other $f \in E(D)$ fix an element $r_f \in H_{e_0}^f$. Since $H_{e_0}^{e_0}$ is an amenable group, by [107, Corollary C] there is a sequence of projections $\{p_n\}_{n \in \mathbb{N}}$ witnessing the quasi-diagonality of $C_r^*(H_{e_0}^{e_0})$. For each pair $e, f \in E(D)$ let $p_n^{e,f} \in \mathcal{B}(\ell^2 S)$ be given by

$$p_n^{e,f} = W_{r_e} V_{r_f} p_n V_{r_f}^* W_{r_e}^*, \tag{6.4}$$

where $W$ is as in Eq. (6.3) above. Note that $p_n = p_n^{e_0,e_0}$, following from the choice of $r_{e_0} = e_0$. It is then clear that $p_n^{e,f}$ are finite dimensional orthogonal projections whose range sits within $\ell^2 H_e^f$, and hence $p_n^{e,f} p_n^{e',f'} = 0$ whenever $(e,f) \neq (e',f')$. In addition, standard computations show that the following commutation relations hold:

$$V_{r_f} p_n^{e_0,e_0} = p_n^{e_0,f} V_{r_f} \quad \text{and} \quad W_{r_e} p_n^{e_0,f} = p_n^{e,f} W_{r_e}. \tag{6.5}$$

In this context, let

$$q_n := \sum_{e,f \in E(D)} p_n^{e,f},$$

which is a finite dimensional orthogonal projection by the above discussion. Moreover, since $p_n$ converges strongly to the identity of $\ell^2 H_{e_0}^{e_0}$, so does $q_n$ converge strongly to the identity of $\ell^2 D$. Thus all we have left to do is prove that $q_n$ asymptotically commutes with every $V_s \in C_r^*(S)$. For this, due to the built-in orthogonality of $p_n^{e,f}$ and $p_n^{e',f'}$:

$$\|V_s\, q_n - q_n\, V_s\| = \left\| \sum_{e,f \in E(D)} V_s\, p_n^{e,f} - p_n^{e,f}\, V_s \right\| = \max_{e \in E(D)} \max_{\substack{f \in E(D) \\ f \leq s^* s}} \left\| V_s\, p_n^{e,f} - p_n^{e,sfs^*}\, V_s \right\| \tag{6.6}$$

where, for the last equality, we have used that $V_s p_n^{e,f} = 0$ unless $f \leq s^* s$. Seen from the point of view of the $\mathcal{H}$-classes themselves this means that $H_e^f \subset D_{s^*s}$ which, by Lemma 6.3.5, is equivalent to $H_e^f \cap D_{s^*s} \neq \varnothing$. By the same Lemma 6.3.5, observe that, in this case, $sH_e^f$ is a (possibly different) $\mathcal{H}$-class in the same $\mathcal{L}$-class as $H_e^f$, and one can check that $sH_e^f = H_e^{sfs^*}$. By Eq. (6.6) it is enough to show that $\|V_s p_n^{e,f} - p_n^{e,sfs^*} V_s\|$ tends to 0 for every ordered pair $e, f \in E(D)$, where $f \leq s^* s$. To this end, observe that

$$\begin{aligned}
\left\| V_s p_n^{e,f} - p_n^{e,sfs^*} V_s \right\| &= \left\| V_s W_{r_e} p_n^{e_0,f} W_{r_e}^* - W_{r_e} p_n^{e_0,sfs^*} W_{r_e}^* V_s \right\| && \text{by (6.4)} \\
&= \left\| W_{r_e} \left( V_s p_n^{e_0,f} - p_n^{e_0,sfs^*} V_s \right) W_{r_e}^* \right\| && \text{as } V_s W_t = W_t V_s \\
&\leq \left\| V_s p_n^{e_0,f} - p_n^{e_0,sfs^*} V_s \right\| && \text{as } W_t \text{ is a partial isometry} \\
&= \left\| \left( V_s p_n^{e_0,f} - p_n^{e_0,sfs^*} V_s \right) V_{r_f} \right\| \\
&= \left\| V_{sr_f} p_n^{e_0,e_0} - p_n^{e_0,sfs^*} V_{sr_f} \right\|, && \text{by (6.5)}
\end{aligned}$$

where, for the prior to last equality, we have used Lemma 6.3.6. Indeed, if $a = V_s p_n^{e_0,f} - p_n^{e_0,sfs^*} V_s$ and $v = V_{r_f}$ then $a(1 - vv^*) = 0$ by a simple computation. Therefore, by renaming $t := sr_f$ note that $t \in H_{e_0}^{sfs^*} = H_{e_0}^{tt^*}$. Indeed:

$$t^* t = (sr_f)^* (sr_f) = r_f^* s^* s r_f = r_f^* s^* s f r_f = r_f^* f r_f = e_0.$$

Furthermore

$$V_{sr_f} p_n^{e_0,e_0} = V_{r_{tt^*}} V_{r_{tt^*}^* t}\, p_n \quad \text{and} \quad p_n^{e_0,sfs^*} V_{sr_f} = p_n^{e_0,tt^*} V_{r_{tt^*}} V_{r_{tt^*}^*} V_t = V_{r_{tt^*}} p_n V_{r_{tt^*}^* t}, \tag{6.7}$$

and therefore the fact that $q_n$ is asymptotically central in $C_r^*(S)$ follows from Eq. (6.7) and the fact that $p_n$ is asymptotically central in $C_r^*(H_{e_0}^{e_0})$:

$$\left\|V_s p_n^{e,f} - p_n^{e,sfs^*} V_s\right\| \leq \left\|V_{r_{tt^*}} \left(V_{r_{tt^*}^* t} p_n - p_n V_{r_{tt^*}^* t}\right)\right\| \leq \left\|V_{r_{tt^*}^* t} p_n - p_n V_{r_{tt^*}^* t}\right\| \xrightarrow{n\to\infty} 0,$$

as $r_{tt^*}^* t \in H_{e_0}^{e_0}$. □

**Theorem 6.3.8.** *Let $S$ be a countable and discrete inverse semigroup. Suppose the following holds:*

*(1) No $\mathcal{D}$-class in $S$ has infinitely many projections.*

*(2) All the maximal subgroups of $S$ are amenable (as groups).*

*Then $C_r^*(S)$ is a quasi-diagonal $C^*$-algebra.*

*Proof.* Let $\mathfrak{D}$ be the, possibly infinite, set of $\mathcal{D}$-classes of $S$. As it is countable we may write it as $\mathfrak{D} = \cup_{n\in\mathbb{N}} \mathfrak{D}_n$, where $\mathfrak{D}_n \subset \mathfrak{D}_{n+1}$ are finite sets. Moreover, write $S = \cup_{n\in\mathbb{N}} \mathcal{F}_n$ as an increasing sequence of finite subsets. Now, for every $n \in \mathbb{N}$ and $D \in \mathfrak{D}_n$, let $\{q_n^D\}_{n\in\mathbb{N}}$ be as in the conclusion of Lemma 6.3.7. In addition, note we may suppose that

$$\max_{s\in\mathcal{F}_n} \max_{D\in\mathfrak{D}_n} \left\|V_s\, q_n^D - q_n^D\, V_s\right\| \leq 1/n. \tag{6.8}$$

for every $n \in \mathbb{N}$. Consider

$$q_n := \sum_{D\in\mathfrak{D}_n} q_n^D.$$

It is clear that $q_n$ is a finite dimensional orthogonal projection. Moreover, since $q_n^D$ strongly converges to $1_D$ for every $\mathcal{D}$-class $D$, it follows that so does $q_n$ strongly converge to 1. Lastly, $q_n$ is also asymptotically central by Eq. (6.8):

$$\begin{aligned} \max_{s\in\mathcal{F}_n} \|V_s q_n - q_n V_s\| &= \max_{s\in\mathcal{F}_n} \left\|\sum_{D\in\mathfrak{D}_n} V_s\, q_n^D - q_n^D\, V_s\right\| \\ &= \max_{s\in\mathcal{F}_n} \max_{D\in\mathfrak{D}_n} \left\|V_s\, q_n^D - q_n^D\, V_s\right\| \leq 1/n \xrightarrow{n\to\infty} 0. \end{aligned}$$

Note that for the second equality we have used that $q_n^D$ and $q_n^{D'}$ are orthogonal projections when $D$ and $D'$ are different $\mathcal{D}$-classes (see (1) in Lemma 6.3.7). □

**Remark 6.3.9.** Observe that both conditions (1) and (2) in Theorem 6.3.8 can be seen geometrically in the undirected graph $\Lambda_S$ constructed in Subsection 5.1.1. Indeed, condition (1) means the Schützenberger graphs of $S$ are finite unions of group Cayley graphs, and hence are coarsely equivalent to their own unique $\mathcal{H}$-class. Moreover, that $\mathcal{H}$-class is amenable by condition (2). Thus the hypothesis of Theorem 6.3.8 state that not only is $S$ amenable, but every $\mathcal{L}$-class of $S$ is (up to coarse equivalence) an amenable group (see Figure 6.2).

In Theorem 6.3.8 (and in Lemma 6.3.7 for that matter) note that the condition saying the $\mathcal{D}$-class $D \subset S$ has finitely many projections is needed, and for various

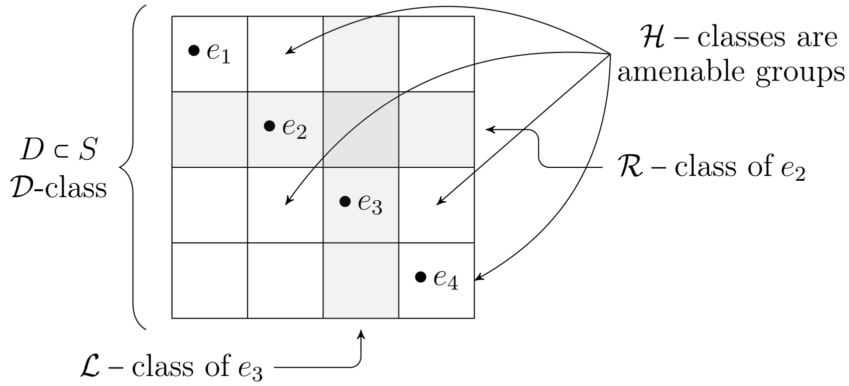


FIGURE 6.2: Visual representation of a $\mathcal{D}$-class $D$ in a semigroup $S$ satisfying conditions (1) and (2) in Theorem 6.3.8.

reasons. Indeed, should we allow $D$ to have infinitely many projections then the bicyclic monoid, see Example 5.1.6, and its unique $\mathcal{D}$-class would meet the hypothesis of the theorem. However $C_r^*(\mathcal{T})$ is not quasi-diagonal, as it is not even finite. Secondly, even if the partial order in a $\mathcal{D}$-class $D$ restricts to equality (which, for the record, does not happen in $\mathcal{T}$) then the techniques in Lemma 6.3.7 cannot be applied. For instance, if $D \cap E(S)$ is infinite then $\sum_{e,f \in E(S) \cap D} p_n^{e,f}$ is infinite dimensional, and cannot be immediately *cut down* in order to produce a finite dimensional projection that almost commutes with $V_s$. Observe, for example, that left multiplication by $s$ may move an $\mathcal{H}$-class $H$ *outside* of any obvious cut down of $p_n$, inevitably violating the almost commuting condition.

# Chapter 7
# Conclusions and open problems

All the results of the thesis have already been proven, and hence this final chapter is dedicated entirely to review some of them, and state some interesting related questions. As is only natural, the language used in this chapter is similar to that of the introductory Chapter 1, i.e., more colloquial and hand-waving than usual. Section 7.1 summarizes some of the most important results of the thesis, most of them answering the questions posed in the Section 1.2, while Section 7.2 exposes some open problems that have remained unanswered during the text.

## 7.1 What we now know...

Chapter 3 answered some questions about amenability in general semigroup theory. It reached the conclusion that there seems to be no satisfactory dichotomy amenable vs. paradoxical in this generality. Even more so, there is no unique notion of Følner sequence, therefore making us consider *strong Følner sequences* and *usual Følner sequences* separately. Finally, as semigroups do not have a canonical left regular representation they cannot be dealt with the C*-techniques here used.

Nevertheless, should one restrict to the particular class of inverse semigroups then the answer is more clear. Indeed, in Chapter 4 we proved (see Theorem 4.1.3) that Day's amenability can be decomposed into two notions, *localization* and *domain-measurability*. Moreover, of these two only one is dynamical, and it can be translated into the tracial property of a state of the crossed product $\ell^\infty(S) \rtimes_r S$ (see Theorem 4.3.7). Actually, the correspondence goes beyond this, since any traces on $\ell^\infty(S) \rtimes_r S$, which are necessarily amenable, can be canonically seen as domain-measures of the semigroup. All of this research places semigroups with algebras and metric spaces in relation with amenability, as opposed to groups. Indeed, dynamically speaking inverse semigroups allow for impossible behavior in the context of groups, such as being an amenable structure containing non-amenable substructures.

The study of $\ell^\infty(S) \rtimes_r S$ and its trace space leads to the consideration of proper and right invariant metrics on the semigroup, with the goal of allowing to see the crossed product as an honest uniform Roe algebra. In Chapter 5 we thus study the Schützenberger graphs of the semigroup, and relate these with the uniform Roe algebra. These remarks lead to the definition of admitting a finite labeling (see Proposition 5.2.18) and, hence, to Theorem 5.2.23, which characterizes when the crossed product can be seen as an honest uniform Roe algebra. Of particular interest is the case when the latter does *not* hold, of course, since then the methods here presented generalize previous methods, such as the ones in [8]. In addition, it is not clear which inverse semigroups admit finite labelings, though it is clear that some do not, the prime example being $(\mathbb{N}, \min)$, see Example 5.2.21. Surprisingly, this FL condition also appeared when considering some quasi-isometric invariant properties, such as domain-measurability (see Theorem 5.3.5) and property A (see Theorem 5.3.10). In particular, Theorem 5.3.10 generalizes the known equivalence between exactness and property A in geometric group theory to our context. Furthermore, the fact that Theorem 5.3.10 requires admitting a finite labeling makes it all the more engaging.

Lastly, we were able to initiate the study of quasi-diagonality of reduced inverse semigroup C*-algebras in Chapter 6. In particular Theorem 6.2.1 proves that any inverse semigroup whose reduced C*-algebra is quasi-diagonal must admit a domain measure. Moreover, Theorem 6.3.8 gives sufficient conditions for the reciprocal, that is, proves that inverse semigroups whose $\mathcal{H}$-classes are amenable (as groups) and with no $\mathcal{D}$-class with infinitely many projections have quasi-diagonal C*-algebras. Of particular interest is the fact that this is only a partial result, since these semigroups are domain-measurable, but we have not been able to prove an *if-and-only-if* theorem.

To summarize, we have obtained a more clear picture of the relation between different approximation notions, such as semigroup amenability, the existence of (amenable) traces, nuclearity, exactness, property A and quasi-diagonality in the context of (inverse) semigroups.

## 7.2 ... and what we still don't

We end the thesis the way this kind of text oughts to end, by discussing some interesting questions that have appeared along the text and have, unfortunately, been left unanswered. Firstly, going back to Chapter 5 and the construction of the graph $\Lambda_S$, note that Lemma 5.1.7 precisely states *right-invariance* in the context of inverse semigroups. Moreover, observe that any (countable and discrete) group can be equipped with a unique (up to coarse equivalence) proper and right invariant metric. These remarks beg the following questions:

(QM.1) *Can we equip an inverse semigroup with a proper and right-invariant metric? Is such a metric unique up to coarse equivalence? If not, is it unique among the metrics arising from generating sets? Is it unique in some particular cases, such as F-inverse semigroups?*

(QM.2) *Given the answers to the above are positive, which results of this thesis translate into that context? Does admitting a finite labeling play a role?*

In the same bag of questions we find the following, which are unknown. In particular, for (QA.2) observe that the answer is known to be *yes* (see Theorem 5.3.10), but the only proof so far uses C*-theory.

(QA.1) *Is there any non-trivial relation between the domain-measurability of a semigroup and its property A?*

(QA.2) *Given an E-unitary inverse semigroup $S$, can one directly translate the property A of $G(S)$ into property A of $S$? That is, is there some coarse geometric argument relating these properties?*

(QA.3) *Is there any class of inverse semigroups whose Schützenberger graphs do not have property A?*

Lastly among the questions related to metric spaces, the so-called *warped cone* construction (see, e.g., [90]) is of interest since it may yield a space without property A. It would be interesting to see whether this construction can happen for an inverse semigroup (note that the HLS-groupoids of Section 5.4.5 do have semigroup models):

(QW.1) *Can the warped cone construction of [90] be replicated in the context of inverse semigroups? That is, is there an inverse semigroup whose associated graph $\Lambda_S$ is the warped cone $\mathcal{O}_\Gamma Y$, associated to an action of a group $\Gamma$ on a metric space $Y$?*

Chapter 6 is, to be honest, rather incomplete. Indeed, note that the answers to the questions of the introducion (see Section 1.2) about quasi-diagonality have, at best, only been partially answered, begging:

(QQ.1) *Solve Rosenberg's conjecture in a satisfactory manner in the context of discrete inverse semigroups, that is, give a characterization of the quasi-diagonality of $C^*_r(S)$ in terms of algebraic conditions on the inverse semigroup $S$.*

(QQ.2) *Is there an explicit solution to Rosenberg's conjecture in the context of residually finite inverse semigroups? What about in the context of discrete amenable groups?*

Observe that for (QQ.1) it would be enough to prove that if $C^*_r(S)$ is quasi-diagonal then no $\mathcal{D}$-class in $S$ has infinitely-many idempotents and every $\mathcal{H}$-class is amenable, since that is the sufficient condition appearing in Theorem 6.3.8. Lastly, note that the second question of (QQ.2) has been open for quite some time (see, for instance, [107, 116, 117]). The last group of questions we wish to mention come from the analogies between groupoids and inverse semigroups, and from the fact that groupoid models for every classifiable C*-algebra exist (see Li's recent work [64]). To keep the following short we name only a couple of the natural questions arising from this point of view.

(QA.1) *Is there any inverse monoid analogue of the Jiang-Su algebra $\mathcal{Z}$?*

(QA.2) *Is there any analogue notion of strongly self-absorbing monoid? Are UHF monoids of infinite type strongly self-absorbing in such sense?*

For (QA.1) observe there is no inverse semigroup $S$ such that $C_r^*(S) \cong \mathcal{Z}$. Indeed, if that were the case then all the non-trivial projections from the spectrum of $S$ would sit within $\mathcal{Z}$, which is well known to be unital projectionless. Therefore (QA.1) is, arguably, rather ambiguous.

As has been said, these are but a few of the questions that have appeared along the way. I hope you have enjoyed reading this.

*Theses are not made to be believed,*
*but to be made subject to inquiry.*
William of Baskerville

# Bibliography


[1] C. Anantharaman-Delaroche and J. Renault, *Amenable groupoids*, L'Enseignement matheématique / Monographie, 36 (2000).

[2] C. Anantharaman-Delaroche, *Exact groupoids*, https://arxiv.org/abs/1605.05117 (2016).

[3] C. Anantharaman-Delaroche, *Some remarks about the weak containment property for groupoids and semigroups*, https://arxiv.org/abs/1604.01724 (2019).

[4] R. Antoine, H. Thiel and F. Perera, *Tensor products and regularity properties of Cuntz semigroups*, Ame. Math. Soc. 2018.

[5] P. Ara and R. Exel, *Dynamical systems associated to separated graphs, graph algebras, and paradoxical decompositions*, Adv. Math. **252** (2014) 748–80

[6] P. Ara and F. Lledó, *Amenable traces and Følner C*-algebras*, Expo. Math. **32** (2014) 161–177.

[7] P. Ara, K. Li, F. Lledó and J. Wu, *Amenability of coarse spaces and* $\mathbb{K}$*-algebras*, Bull. Math. Sci. **8** (2018) 257–306.

[8] P. Ara, K. Li, F. Lledó and J. Wu, *Amenability and uniform Roe algebras*, J. Math. Anal. and Appl. **459** (2018) 686–716.

[9] P. Ara, F. Lledó and D. Martínez, *Amenability and paradoxicality in semigroups and C*-algebras*, J. Func. Anal. **279** (2020).

[10] L. N. Argabright and C. O. Wilde *Semigroups satisfying a strong Følner condition*, Proc. Ame. Math. Soc. **17** (1967) 587–591.

[11] C. J. Ash and T. E. Hall, *Inverse semigroups on graphs*, Semigroup Forum **11** (1975) 140-–145.

[12] M. Aukhadiev, *An inverse semigroup approach to the C*-algebras and crossed products of cancellative semigroups*, J. Noncomm. Geom. **12** (2018), 693–731.

[13] L. Bartholdi, *On amenability of group algebras, I*, Israel J. Math. **168** (2008) 153–165.

[14] E. Bédos, *Notes on hypertraces and C*-algebras*, J. Op. Th. **34** (1995) 285–306.

[15] E. Bédos, *On Fölner nets, Szegö's theorem and other eigenvalue distribution theorems*, Expo. Math. **15** (1997) 193–228.

[16] J. Block and S. Weinberger, *Aperiodic Tilings, Positive Scalar Curvature, and Amenability of Spaces*, J. Amer. Math. Soc. **5** (1992) 907–918.

[17] C. Bönicke and K. Li, *Ideal structure and pure infiniteness of ample groupoid C*-algebras*, Erg. Th. and Dyn. Syst. **40** (2020) 34–63.

[18] B.M. Braga, I. Farah and A. Vignati, *General uniform Roe algebra rigidity*, preprint 2020, arXiv: 2001.10266.

[19] N. P. Brown, *Invariant Means and Finite Representation Theory of C*-algebras*, Mem. Amer. Math. Soc. **184** (2006), viii+105.

[20] N. P. Brown and N. Ozawa, *C*-Algebras and finite-dimensional approximations*, American Mathematical Society (2008).

[21] T. Ceccherini-Silberstein, R. Grigorchuk and P. de la Harpe, *Amenability and paradoxical decompositions for pseudogroups and for discrete metric spaces*, Proc. Steklov Inst. Math. **224** (1999) 57–97. (Updated version `http://arxiv.org/abs/1603.04212`).

[22] T. Ceccherini-Silberstein and A.Y. Samet-Vaillant, *Gromov's translation algebras, growth and amenability of operator algebras*, Expo. Math. **26** (2008) 141–162.

[23] T. Ceccherini-Silberstein and A. Y. Samet-Vaillant, *Celullar automata and groups*, Springer Verlag, 2010.

[24] X. Chen, R. Tessera, X. Wang, and G. Yu, *Metric sparsification and operator norm localization*, Adv. Math. **218** (2008) 1496–1511.

[25] J. Cohen and A. Figà-Talamanca, *Idempotents in the reduced C*-algebra of a free group*, Proc. Amer. Math. Soc. **103** (1988) 779–782.

[26] A. Connes, *Classification of injective factors. Cases* $II_1$, $II_\infty$, $III_\lambda$, $\lambda \neq 1$, Ann. Math. **104** (1976) 73–115.

[27] J. Cuntz, *Simple C*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977) 173–185.

[28] K. R. Davidson, *C*-algebras by Example*, American Mathematical Society (1996).

[29] M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957) 509–544.

[30] J. Deprez, *Fair amenability for semigroups*, J. of Algebra **459** (2016) 350–375.

[31] J. Duncan and I. Namioka, *Amenability of inverse semigroups and their semigroup algebras*, Proc. Royal Soc. of Edinburgh Sect. A **10** (1978) 309–321.

[32] J. Duncan and A. L. T. Paterson, *C*-algebras of inverse semigroups*, Proc. Edinburgh Math. Soc. **28** (1985) 41–58.

[33] S. Eilenberg, *Automata, Languages and Machines*, Volume A. New York: Academic Press (1974).

[34] G. Elek, *The amenability of affine algebras*, J. Algebra **264** (2003) 469–478.

[35] R. Exel, *Partial actions of groups and actions of inverse semigroups*, Proc. Amer. Math. Soc. **126** (1998) 3481–3494.

[36] R. Exel, *Inverse semigroups and combinatorial C*-algebras*, Bull. Brazil. Math. Soc. (N.S.) **39** (2008) 191–313.

[37] R. Exel and C. Starling, *Amenable actions of inverse semigroups*, Erg. Th. and Dyn. Syst. **37** (2017) 481–489.

[38] M. Finn-Sell, *Inverse semigroups in coarse geometry*, PhD Thesis, Univ. Southampton (2013).

[39] E. Følner, *On groups with full Banach mean value*, Math. Scand. **3** (1955) 243–254.

[40] J. Gabe, *Quasidiagonal traces on exact C*-algebras*, J. Func. Anal. **272** (2017) 1104–1120.

[41] R. Gray and M. Kambites, *Groups acting on semimetric spaces and quasi-isometries of monoids*, Trans. Ame. Math. Soc. **365** (2013) 555-578.

[42] R. Gray and M. Kambites, *Amenability and geometry of semigroups*, Trans. Ame. Math. Soc. **369** (2017) 8087–8103.

[43] J. A. Green, *On the structure of semigroups*, Annals of Math. **54** (1951) 163–172.

[44] M. Gromov, *Topological Invariants of Dynamical Systems and Spaces of Holomorphic Maps: I*, Math. Phys. Anal. Geom. **2** (1999) 323–415.

[45] M. Gromov, *Random walk in random groups*, Geom. Funct. Anal. **13** (2003) 73–146.

[46] D. Hadwin, *Strongly quasidiagonal C*-algebras*, J. Op. Th. **18** (1987) 3–18 (appendix by J. Rosenberg).

[47] P. R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76** (1970) 887–933.

[48] N. Higson, V. Lafforgue, and G. Skandalis, *Counterexamples to the Baum-Connes conjecture*, Geom. Funct. Anal. **12** (2002) 330–354.

[49] C. Hollings, *Three approaches to inverse semigroups*, Eur. J. Pure Appl. Math. **8** (2015) 294–323.

[50] J. M. Howie, *An introduction to semigroup theory*, Academic Press, London (1976).

[51] J. M. Howie, *Fundamentals of Semigroup Theory*, Oxford University Press (1995).

[52] D. Kerr and H. Li, *Ergodic Theory. Independence and Dichotomies*, Springer Verlag, 2017.

[53] A. Khukhro, *Box spaces, group extensions and coarse embeddings into Hilbert space*, J. Funct. Anal. **263** (2012) 115–128.

[54] E. Kirchberg and C. Phillips, *Embedding of exact C*-algebras in the Cuntz algebra* $\mathcal{O}_2$, J. Reine Angew. Math. **525** (2000) 17–53.

[55] M. Klawe, *Semidirect product of semigroups in relation to amenability, cancellation properties, and strong Følner conditions*, Pac. J. Math. **73** (1977) 91–106.

[56] G. Kudryavtseva, M.V. Lawson, D. Lenz and P. Resende, *Invariant means on Boolean inverse monoids*, Semigroup Forum **92** (2016) 77–101.

[57] A. Kumjian, *On localizations and simple C*-algebras*, Pac. J. Math. **112** (1984) 141–192.

[58] M. V. Lawson, *Inverse semigroups: the Theory of Partial Symmetries*, World Scientific. (1998).

[59] M. V. Lawson and P. Scott, *AF inverse monoids and the structure of countable MV-algebras*, J. Pure and App. Alg. **221** (2017) 45–74.

[60] M. V. Lawson, *Recent developments in inverse semigroup theory*, Semigroup Forum (2019), https://doi.org/10.1007/s00233-019-10067-4.

[61] F. Lledó and D. Martínez, *The uniform Roe algebra of an inverse semigroup*, (2020) submitted https://arxiv.org/abs/2004.01890.

[62] X. Li, *Semigroup C*-algebras and amenability of semigroups*, J. Func. Anal. **262** (2012) 4302–4340.

[63] X. Li, *Nuclearity of semigroup C*-algebras and the connection to amenability*, Adv. Math. **244** (2013) 626–662.

[64] X. Li, *Every classifiable simple C*-algebra has a Cartan subalgebra*, Inv. Math. **219** (2020) 653–699.

[65] K. Li and R. Willett, *Low dimensional properties of uniform Roe algebras*, J. London Math. Soc. **97** (2018) 98–124.

[66] J. Meakin and N. Szakács, *Inverse monoids and immersions of 2-complexes*, Inter. J. Alg. Comput. **25** (2015) 301–323.

[67] D. Milan, *C*-algebras of inverse semigroups: amenability and weak containment*, J. Operator Theory **63** (2010) 317–332.

[68] D. Milan and B. Steinberg, *On inverse semigroup C*-algebras and crossed products*, Groups Geom. Dyn. **2** (2014) 485–512.

[69] G. J. Murphy, *Ordered groups and crossed products of C*-algebras*, Pacific J. Math. **2** (1991) 319–349.

[70] G. J. Murphy, *Crossed products of C*-algebras by semigroups of automorphisms*, Proc. Lond. Math. Soc. **3** (1994) 423–448.

[71] I. Namioka, *Følner's conditions for amenable semi-groups*, Math. Scand. **15** (1964) 18–28.

[72] A. Nica, *C*-algebras generated by isometries and Wiener–Hopf operators*, J. Op. Th. **27** (1992) 17–52.

[73] J. von Neumann, *Zur allgemeinen Theorie des Masses*, Fund. Math. **13** (1929) 73–116.

[74] M. Nivat and J. F. Perrot, *Une généralisation du monoïde bicyclique*, Comptes Rendus de l'Académie des Sciences de Paris. **271** (1970) 824-–827.

[75] P. W. Nowak, *Coarsely embeddable metric spaces without Property A*, J. Funct. Anal. **252** (2007) 126–136.

[76] P. W. Nowak and G. W. Yu, *Large Scale Geometry*, European Mathematical Society. (2012).

[77] S. Orfanos, *Quasidiagonality of crossed products*, J. Op. Th. **66** (2011) 209–216.

[78] D. Osajda, *Small cancellation labellings of some infinite graphs and applications*, Acta Math. **225** (2021) 159–191.

[79] N. Ozawa, *Amenable actions and exactness for discrete groups*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000) 691–695.

[80] A. L. T. Paterson, *Weak containment and Clifford semigroups*, Proc. Roy. Soc. Edinburgh Sect. A. **81** (1978) 23–30.

[81] A. L. T. Paterson, *Groupoids, Inverse Semigroups, and their Operator Algebras*, Springer Verlag (1999).

[82] A.L.T. Paterson, *Amenability*, American Mathematical Society (1988).

[83] G. B. Preston, *Inverse semi-groups*, J. London Math. Socs. 29 (1954) 396-–403.

[84] G. B. Preston, *Inverse semi-groups with minimal right ideals*, J. London Math. Soc. **29** (1954) 404–411.

[85] G. B. Preston, *Representations of inverse semi-groups*, J. London Math. Soc. **29** (1954) 411–9.

[86] T. Rainone, *Finiteness and Paradoxical Decompostions in C*-Dynamical Systems*, J. Noncommut. Geom. **11** (2017) 791–822.

[87] T. Rainone and A. Sims, *A dichotomy for groupoid C*-algebras*, Erg. Theory and Dyn. Syst. **40** (2019) 521–563.

[88] J. Renault, *A Groupoid Approach to C*-algebras*, Springer Verlag (1980).

[89] J. Roe, *Lectures on coarse geometry*, American Mathematical Society, (2003).

[90] J. Roe, *Warped cones and property A*, Geom. & Top. **9** (2005) 163–178.

[91] M. Rørdam and A. Sierakowski, *Purely infinite C*-algebras arising from crossed products*, Erg. Th. and Dyn. Sys. **32** (2012) 273–293.

[92] V. Runde, *Lectures on amenability*, Springer Verlag (2002).

[93] H. Sako, *Property A and the operator norm localization property for discrete metric spaces*, J. Reine Angew. Math. **690** (2014) 207–216.

[94] H. Sako, *Finite dimensional approximation properties for uniform Roe algebras*, J. London Math. Soc. https://doi.org/10.1112/jlms.12330 (2020).

[95] E. Scarparo, *Characterizations of locally finite actions of groups on sets*, Glasgow Math. J. **60** (2018) 285–288.

[96] C. Schafhauser, *A new proof of the Tikuisis-White-Winter Theorem*, to appear in J. Reine Angew. Math. (2017) 14 pages.

[97] C. Schafhauser, *Subalgebras of simple AF-algebras*, to appear in Ann. of Math. **2** (2018) 34 pages.

[98] M. P. Schützenberger, *$\mathcal{D}$-représentation des demi-groupes*, C. R. Acad. Sci. Paris. **244** (1957) 1994–1996.

[99] M. P. Schützenberger, *Sur la représentation monomiale des demi-groupes*, C. R. Acad. Sci. Paris. **246** (1958) 865–867.

[100] G. Skandalis, J. L. Tu and G. Yu, *The coarse Baum-Connes conjecture and groupoids*, Topology **41** (2002) 807–834.

[101] N. Sieben, *C*-crossed products by partial actions and actions of inverse semigroups*, J. Austral. Math. Soc. (Series A). **63** (1997) 32–46.

[102] J.R. Sorenson, *Existence of measures that are invariant under a semigroup of transformations*, Ph.D. Thesis, (1967).

[103] J. Špakula and R. Willett, *On rigidity of Roe algebras*, Adv. Math. **249** (2013) 289–310.

[104] C. Starling, *C*-algebras of Boolean inverse monoids–traces and invariant means*, Doc. Math. **21** (2016) 809–840.

[105] J. B. Stephen, *Presentations of inverse monoids*, J. Pure and App. Alg. **6** (1990) 81–112.

[106] A. Tarski, *Algebraische Fassung des Maßproblems*, Fund. Math. **31** (1938) 207–223.

[107] A. Tikuisis, S. White and W. Winter, *Quasidiagonality of nuclear C*-algebras*, Ann. Math. **185** (2017) 229–284.

[108] G. Tomkowicz and S. Wagon, *The Banach-Tarski Paradox (second edition)*, Cambridge University Press (2016).

[109] V. V. Wagner, *Generalised groups*, Proc. USSR Ac. Scn. **84** (1952) 1119–1122.

[110] V. V. Wagner, *The theory of generalized heaps and generalized groups*, Sb. Mat. **3** (1953) 545–632.

[111] D. Voiculescu, *A note on quasi-diagonal C*-algebras and homotopy*, Duke Math. J. **62** (1991) 267–271.

[112] D. Voiculescu, *Around quasidiagonal operators*, Integr. Equat. Op. Th. **17** (1993) 137–149.

[113] R. Willett, *Some notes on property A*, Limits of graphs in group theory and computer science, EPFL Press, Lausanne (2009) 191–281.

[114] R. Willett, *Property and graphs with large girth*, J. Topol. Anal. **3** (2011) 377–384.

[115] R. Willett, *A non-amenable groupoid whose maximal and reduces C*-algebras are the same*, Münster J. Math. **8** (2015) 241–252.

[116] W. Winter *QDQ vs UCT*, Abel Symposia **12** (2016) 321–342.

[117] W. Winter *Structure of nuclear C*-algebras: from quasi-diagonality to classification, and back again.* Proc. ICM 2018 **2** (2017) 1797–1820.

[118] Z. Yang, *Følner numbers and Følner type conditions for amenable semigroups*, Illinois J. Math. **31** (1987) 496–517.

[119] G. Yu, *The coarse Baum–Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Inv. Math. **139** (2000) 201–240.

# List of topics

# List of symbols

| | |
|---|---|
| $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ | Non-negative integers, integers, real and complex numbers |
| $\mathbb{F}_n$ | Free non-abelian group of rank $n$ |
| $\sqcup$ | Disjoint union of sets |
| | **Functional analysis notation** |
| $\|\|a\|\|$ | Operator norm of $a$ |
| $\|\|a\|\|_2$ | Hilbert-Schmidt norm of $a$ |
| $\ell^\infty(X)$ | Bounded functions on a discrete set $X$ |
| $M_n$ | Full matrix algebra of size $n$ |
| $1_n$ | Identity matrix of size $n \times n$ |
| $\mathrm{tr}_n$ | Normalized tracial state of $M_n$ |
| $\mathcal{O}_n$ | Cuntz algebra on $n$ generators |
| $\mathcal{Q} = \otimes_{n\in\mathbb{N}} M_n$ | Universal UHF algebra |
| $\mathcal{B}(\mathcal{H})$ | Algebra of bounded linear operators on $\mathcal{H}$ |
| $\mathcal{K}$ | Algebra of compact operators on $\ell^2$ |
| c.(u.)c.p. | contractive (unital) and completely positive |
| $A \otimes B$ | Tensor product of $A$ and $B$ |
| | **Extended metric space notation** |
| $(X, d)$ | Extended metric space with metric $d$ |
| $\partial_r A$ | $r$-boundary of a set $A$ |
| $\mathcal{N}_r A$ | $r$-neighborhood of a set $A$ |
| $B_r(x)$ | Ball of radius $r$ centered on $x$ |
| $\ell^2 X$ | Hilbert space of square-summable functions $f\colon X \to \mathbb{C}$ |
| $C^*_{u,alg}(X, d)$ | Operators of bounded propagation in $\mathcal{B}(\ell^2 X)$ |
| $C^*_u(X, d)$ | Uniform Roe algebra of $(X, d)$, norm closure of $C_{u,alg}(X, d)$ |
| | **Semigroup notation** |
| $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ | Green's relations |
| $s^{-1}A$ | Preimage of the set $A$ under $s$ |
| $\ell(s)$ | Length of the element $s \in S$ |
| $\sim$ | Equidecomposability relation between two sets |
| $\mathrm{Cay}(S, K)$ | Left Cayley graph of $S$ with respect to $K$ |
| $\Lambda(L, K) = \Lambda_L$ | Schützenberger graph of $\mathcal{L}$-class $L \subset S$ |
| $\Lambda(S, K) = \Lambda_S$ | Disjoint union of Schützenberger graphs of $\mathcal{L}$-classes of $S$ |
| $\mathcal{I}(X)$ | Full symmetric monoid on $X$ |
| $\mathcal{P}_n$ | Polycyclic monoid of rank $n$ |

| | |
|---|---|
| $S_G$ | Box space semigroup construction from a group $G$ |
| | **Semigroup C*-algebras notation** |
| $\mathbb{C}S$ | Complex semigroup algebra |
| $C_r^*(S)$ | Semigroup reduced C*-algebra |
| $\mathcal{R}_{X,alg}$ | Sub-algebra of $\mathcal{B}(\ell^2(S))$ spanned by terms of the form $fV_s$, where $s \in S$ and $f \in \ell^\infty(S)$ |
| $\mathcal{R}_X = \ell^\infty(X) \rtimes_r S$ | Reduced semigroup crossed product, norm closure of $\mathcal{R}_{X,alg}$ |
| $\mathcal{E}: \mathcal{B}(\ell^2 X) \to \ell^\infty(X)$ | Canonical conditional expectation |
| $v: S \to \mathcal{I}(S)$ | Wagner-Preston representation of an inverse semigroup |
| $D_{s^*s} = s^*sS$ | Domain of $s \in S$ under $v$ |
| $E_{s^*s}$ | Bounded functions on $S$ with support contained in $D_{s^*s}$ |
| $\mathrm{Typ}(\alpha)$ | Type semigroup associated to the representation $\alpha$ |
| $V: S \to \mathcal{B}(\ell^2 X)$ | (Left) regular representation of $S$ associated to $\alpha: S \to \mathcal{I}(X)$ |
| $W: S \to \mathcal{B}(\ell^2 S)$ | Right regular representation of $S$ |
| $G(S)$ | Maximal group homomorphic image of $S$ |
| $\sigma: S \to G(S)$ | Canonical quotient map |
| | **Universal groupoid construction** |
| $\hat{E}_0$ | Space of filters of $E(S)$ |
| $D^\theta_{s^*s}$ | Compact open set of filters containing $s^*s$ |
| $\theta_s: D^\theta_{s^*s} \to D^\theta_{ss^*}$ | Canonical action on the set of filters |
| $[s,\zeta]$ | Germ of the pair $(s,\zeta)$, where $s \in S$ and $\zeta \in D^\theta_{s^*s}$ |
| $G_{\mathcal{U}}(S)$ | Paterson's universal groupoid associated to $S$ |