Amenability and coarse geometry of (inverse) semigroups and C*-algebras

Diego Martínez

March 5th, 2021

Universidad Carlos III de Madrid - Instituto de Ciencias Matemáticas
lumartin@math.uc3m.es

Thesis (e-)defense

Outline

1. Main schemes and questions

Pyramid schemes and paradoxical behaviors

Do Ponzi schemes exist?

Pyramid schemes and paradoxical behaviors

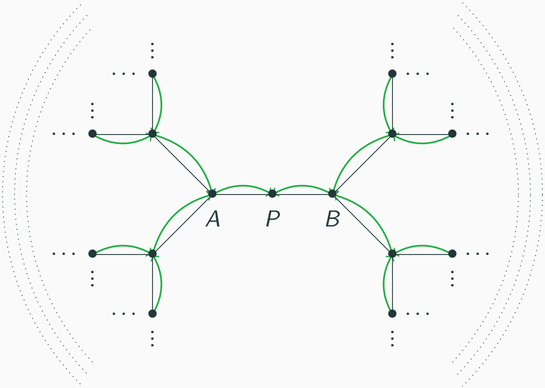
Do Ponzi schemes exist?

- P asks A and B, these give P 1\$; and so on; and on...
- Everybody gains money in a **finite time** by moving it around
- Needs infinitely many people

Pyramid schemes and paradoxical behaviors

Do Ponzi schemes exist?

- P asks A and B, these give P 1\$; and so on; and on...
- Everybody gains money in a **finite time** by moving it around
- Needs infinitely many people *in a certain form*:



Groups: von Neuman, Tarski and Følner

Let G be a countable and discrete group:

- G is *amenable* if there is a prob. measure $\mu\colon G \to [0,1]$ such that $\mu(g^{-1}A) = \mu(A)$ for all $g \in G$ and $A \subset G$
- G satisfies the *Følner condition* if for every $\varepsilon > 0$ and $K \Subset G$ there is $F \Subset G$ such that $|KF \cup F| \leq (1+\varepsilon)|F|$
- G is *paradoxical* if there are $A_i, B_j \subset G$ and $a_i, b_j \in G$ with $G = \sqcup_{i=1}^{n} a_i A_i = \sqcup_{j=1}^{m} b_j B_j \supset (\sqcup_{i=1}^{n} A_i) \sqcup \left(\sqcup_{j=1}^{m} B_j\right)$

Groups: von Neuman, Tarski and Følner

Let G be a countable and discrete group:

- G is *amenable* if there is a prob. measure $\mu\colon G \to [0,1]$ such that $\mu(g^{-1}A) = \mu(A)$ for all $g \in G$ and $A \subset G$
- G satisfies the *Følner condition* if for every $\varepsilon > 0$ and $K \Subset G$ there is $F \Subset G$ such that $|KF \cup F| \leq (1+\varepsilon)|F|$
- G is *paradoxical* if there are $A_i, B_j \subset G$ and $a_i, b_j \in G$ with $G = \sqcup_{i=1}^{n} a_i A_i = \sqcup_{j=1}^{m} b_j B_j \supset (\sqcup_{i=1}^{n} A_i) \sqcup \left(\sqcup_{j=1}^{m} B_j\right)$

Theorem (von Neumann 1927, Tarski 1929, Følner 1956)

G is amenable $\Leftrightarrow$ G satisfies the Følner condition
$\Leftrightarrow$ G is not paradoxical

We call these equivalences *classical*

Groups: von Neuman, Tarski and Følner

Let G be a countable and discrete group:

- G is *amenable* if there is a prob. measure $\mu\colon G \to [0,1]$ such that $\mu(g^{-1}A) = \mu(A)$ for all $g \in G$ and $A \subset G$
- G satisfies the *Følner condition* if for every $\varepsilon > 0$ and $K \Subset G$ there is $F \Subset G$ such that $|KF \cup F| \leq (1+\varepsilon)|F|$
- G is *paradoxical* if there are $A_i, B_j \subset G$ and $a_i, b_j \in G$ with $G = \sqcup_{i=1}^{n} a_i A_i = \sqcup_{j=1}^{m} b_j B_j \supset (\sqcup_{i=1}^{n} A_i) \sqcup \left(\sqcup_{j=1}^{m} B_j\right)$

Theorem (von Neumann 1927, Tarski 1929, Følner 1956)

G is amenable $\Leftrightarrow$ G satisfies the Følner condition
$\Leftrightarrow$ G is not paradoxical

We call these equivalences *classical*

First goal: extend these to inverse semigroups

2. Inverse semigroups and Day's amenability

Inverse semigroups and Wagner-Preston

Definition (Wagner 1952 and Preston 1954)

S is an *inverse semigroup* if for every $s \in S$ there is a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$

Inverse semigroups and Wagner-Preston

Definition (Wagner 1952 and Preston 1954)

S is an *inverse semigroup* if for every $s \in S$ there is a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$

Remarks:

- *Bicyclic monoid:* $\mathcal{B} = \langle a, a^* \mid a^*a = 1 \rangle = \left\{ a^i a^{*j} \mid i, j \geq 0 \right\}$

Definition (Wagner 1952 and Preston 1954)

S is an *inverse semigroup* if for every $s \in S$ there is a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$

Remarks:

- *Bicyclic monoid:* $\mathcal{B} = \langle a, a^* \mid a^*a = 1 \rangle = \left\{ a^i a^{*j} \mid i, j \geq 0 \right\}$
- $E(S) = \left\{ e \in S \mid e^2 = e \right\} = \{ s^*s \mid s \in S \}$ is commutative
- $D_{s^*s} := \{ x \in S \mid x = s^*sx \} = s^*s \cdot S$ is the *domain of* s^*s
- $s \colon D_{s^*s} \to D_{ss^*}$, where $x \mapsto sx$ is a bijection

Inverse semigroups and Wagner-Preston

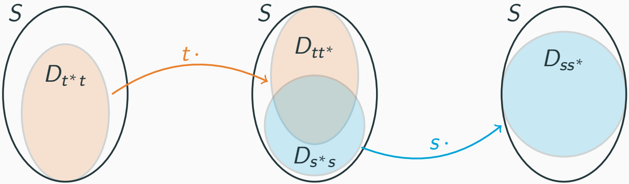
Definition (Wagner 1952 and Preston 1954)

S is an *inverse semigroup* if for every $s \in S$ there is a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$

Remarks:

- *Bicyclic monoid:* $\mathcal{B} = \langle a, a^* \mid a^*a = 1 \rangle = \{a^i a^{*j} \mid i, j \geq 0\}$
- $E(S) = \{e \in S \mid e^2 = e\} = \{s^*s \mid s \in S\}$ is commutative
- $D_{s^*s} := \{x \in S \mid x = s^*sx\} = s^*s \cdot S$ is the *domain of* s^*s
- $s\colon D_{s^*s} \to D_{ss^*}$, where $x \mapsto sx$ is a bijection

Induces the *Wagner-Preston* representation $v\colon S \to \mathcal{I}(S)$:



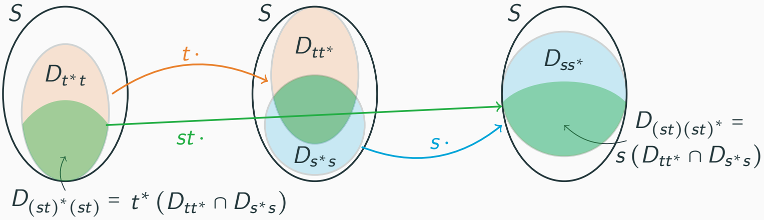
Definition (Wagner 1952 and Preston 1954)

S is an *inverse semigroup* if for every $s \in S$ there is a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$

Remarks:

- *Bicyclic monoid:* $\mathcal{B} = \langle a, a^* \mid a^*a = 1 \rangle = \left\{ a^i a^{*j} \mid i, j \geq 0 \right\}$
- $E(S) = \left\{ e \in S \mid e^2 = e \right\} = \{ s^*s \mid s \in S \}$ is commutative
- $D_{s^*s} := \{ x \in S \mid x = s^*sx \} = s^*s \cdot S$ is the *domain of s^*s*
- $s \colon D_{s^*s} \to D_{ss^*}$, where $x \mapsto sx$ is a bijection

Induces the *Wagner-Preston* representation $v \colon S \to \mathcal{I}(S)$:



Day's amenability

Key idea: multiplication is not injective

s^* is only the inverse of s up to $s^*s \rightsquigarrow$ *local* inverse

Day's amenability

Key idea: multiplication is <u>not</u> injective

s^* is only the inverse of s up to $s^*s \rightsquigarrow$ *local* inverse

$\underline{s^{-1}A := \{x \in S \mid sx \in A\}}$ = pre-image of A

Day's amenability

Key idea: multiplication is <u>not</u> injective

s^* is only the inverse of s up to $s^*s \rightsquigarrow$ *local* inverse

$$\underline{s^{-1}A := \{x \in S \mid sx \in A\}} \;=\; \text{pre-image of } A$$

Definition (Day - 1957, for general semigroups)

S is *amenable* if there is a prob. measure $\mu\colon \mathcal{P}(S) \to [0,1]$ such that $\mu\left(s^{-1}A\right) = \mu(A)$ for every $s \in S$ and $A \subset S$

Examples: amenable groups, and

- $0 \in S \rightsquigarrow \mu(\{0\}) = 1$.
- $\mathbb{F}_2 \sqcup \{1\}$ is not amenable.

Day's amenability

Key idea: multiplication is <u>not</u> injective

s^* is only the inverse of s up to $s^*s \rightsquigarrow$ *local* inverse

$\underline{s^{-1}A := \{x \in S \mid sx \in A\}}$ = pre-image of A

Definition (Day - 1957, for general semigroups)

S is <u>*amenable*</u> if there is a prob. measure $\mu\colon \mathcal{P}(S) \to [0,1]$ such that $\mu\left(s^{-1}A\right) = \mu(A)$ for every $s \in S$ and $A \subset S$

Examples: amenable groups, and

- $0 \in S \rightsquigarrow \mu(\{0\}) = 1$.
- $\mathbb{F}_2 \sqcup \{1\}$ is not amenable.

Remarks:

- Day follows closely von Neumann's steps
- This definition is **irksome**, as it involves pre-images

2. Inverse semigroups and Day's amenability

2.1. Day's amenability split

Day's amenability framed in the inverse case

Pick S and μ invariant measure $\left(\mu(A) = \mu\left(s^{-1}A\right)\right)$. Then:

Day's amenability framed in the inverse case

Pick S and μ invariant measure $\left(\mu(A) = \mu\left(s^{-1}A\right)\right)$. Then:

(1) $\mu(B)$

Day's amenability framed in the inverse case

Pick S and μ invariant measure $\left(\mu(A) = \mu\left(s^{-1}A\right)\right)$. Then:

(1) $\mu(B) = \mu(B \cap D_{ss^*}) + \mu(B \setminus D_{ss^*})$

Day's amenability framed in the inverse case

Pick S and μ invariant measure $\left(\mu(A) = \mu\left(s^{-1}A\right)\right)$. Then:

(1) $\mu(B) = \mu(B \cap D_{ss^*}) + \mu\left(s^{-1}(B \setminus D_{ss^*})\right)$

Day's amenability framed in the inverse case

Pick S and μ invariant measure $\left(\mu(A) = \mu\left(s^{-1}A\right)\right)$. Then:

(1) $\mu(B) = \mu(B \cap D_{ss^*}) + \mu\left(s^{-1}(B \setminus D_{ss^*})\right) = \mu(B \cap D_{ss^*})$

Day's amenability framed in the inverse case

Pick S and μ invariant measure $\left(\mu(A) = \mu\left(s^{-1}A\right)\right)$. Then:

(1) $\mu(B) = \mu(B \cap D_{ss^*}) + \mu\left(s^{-1}(B \setminus D_{ss^*})\right) = \mu(B \cap D_{ss^*})$

(2) If $A \subset D_{ss^*}$ then
$\mu(A)$

Day's amenability framed in the inverse case

Pick S and μ invariant measure $\left(\mu(A) = \mu\left(s^{-1}A\right)\right)$. Then:

(1) $\mu(B) = \mu(B \cap D_{ss^*}) + \mu\left(s^{-1}(B \setminus D_{ss^*})\right) = \mu(B \cap D_{ss^*})$

(2) If $A \subset D_{ss^*}$ then
$\mu(A) = \mu\left(s^{-1}A\right)$

Day's amenability framed in the inverse case

Pick S and μ invariant measure $\left(\mu(A) = \mu\left(s^{-1}A\right)\right)$. Then:

(1) $\mu(B) = \mu(B \cap D_{ss^*}) + \mu\left(s^{-1}(B \setminus D_{ss^*})\right) = \mu(B \cap D_{ss^*})$

(2) If $A \subset D_{ss^*}$ then
$\mu(A) = \mu\left(s^{-1}A\right) = \mu(s^*A) + \mu\left(s^{-1}A \setminus s^*A\right)$

Day's amenability framed in the inverse case

Pick S and μ invariant measure $\left(\mu(A) = \mu\left(s^{-1}A\right)\right)$. Then:

(1) $\mu(B) = \mu(B \cap D_{ss^*}) + \mu\left(s^{-1}(B \setminus D_{ss^*})\right) = \mu(B \cap D_{ss^*})$

(2) If $A \subset D_{ss^*}$ then
$\mu(A) = \mu\left(s^{-1}A\right) = \mu(s^*A) + \mu\left((s^*)^{-1}\left(s^{-1}A \setminus s^*A\right)\right)$

Day's amenability framed in the inverse case

Pick S and μ invariant measure $\left(\mu(A) = \mu\left(s^{-1}A\right)\right)$. Then:

(1) $\mu(B) = \mu(B \cap D_{ss^*}) + \mu\left(s^{-1}(B \setminus D_{ss^*})\right) = \mu(B \cap D_{ss^*})$

(2) If $A \subset D_{ss^*}$ then

$$\begin{aligned}\mu(A) = \mu\left(s^{-1}A\right) &= \mu(s^*A) + \mu\left((s^*)^{-1}\left(s^{-1}A \setminus s^*A\right)\right)\\ &= \mu(s^*A)\end{aligned}$$

Day's amenability framed in the inverse case

Pick S and μ invariant measure $\left(\mu(A) = \mu\left(s^{-1}A\right)\right)$. Then:

(1) $\mu(B) = \mu(B \cap D_{ss^*}) + \mu\left(s^{-1}(B \setminus D_{ss^*})\right) = \mu(B \cap D_{ss^*})$

(2) If $A \subset D_{ss^*}$ then

$$\begin{aligned} \mu(A) = \mu\left(s^{-1}A\right) &= \mu(s^*A) + \mu\left((s^*)^{-1}\left(s^{-1}A \setminus s^*A\right)\right) \\ &= \mu(s^*A) \end{aligned}$$

Theorem (Ara, Lledó, M. - 2020)

μ is invariant $\Leftrightarrow$ both following conditions are satisfied:

(1) *Localization*: $\mu(B) = \mu(B \cap D_{s^*s})$ for all $s \in S, B \subset S$.

(2) *Domain-measure*: $\mu(A) = \mu(sA)$ for all $A \subset D_{s^*s}$.

Domain-measurable semigroups

Examples of domain-measurable semigroups:

- All amenable semigroups.
- Non-ame. & domain-measurable: $S = \mathbb{F}_2 \sqcup \{1\}$ and $\mu = \delta_1$.

Domain-measurable semigroups

Examples of domain-measurable semigroups:

- All amenable semigroups.
- Non-ame. & domain-measurable: $S = \mathbb{F}_2 \sqcup \{1\}$ and $\mu = \delta_1$.

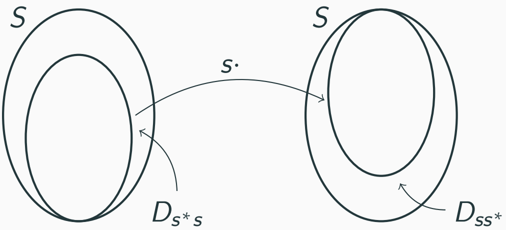
Localization and domain-measurability explanation:

Domain-measurable semigroups

Examples of domain-measurable semigroups:

- All amenable semigroups.
- Non-ame. & domain-measurable: $S = \mathbb{F}_2 \sqcup \{1\}$ and $\mu = \delta_1$.

Localization and domain-measurability explanation:

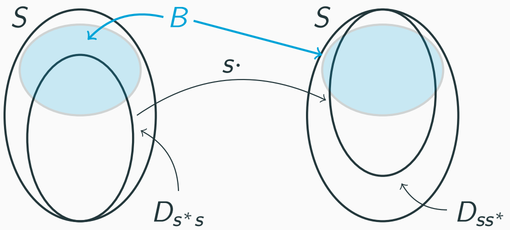


Domain-measurable semigroups

Examples of domain-measurable semigroups:

- All amenable semigroups.
- Non-ame. & domain-measurable: $S = \mathbb{F}_2 \sqcup \{1\}$ and $\mu = \delta_1$.

Localization and domain-measurability explanation:

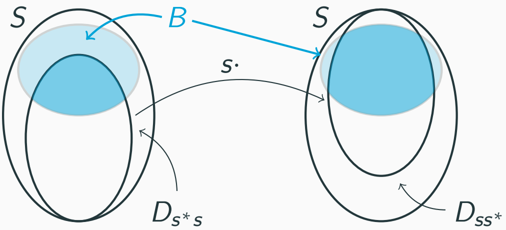


Domain-measurable semigroups

Examples of domain-measurable semigroups:

- All amenable semigroups.
- Non-ame. & domain-measurable: $S = \mathbb{F}_2 \sqcup \{1\}$ and $\mu = \delta_1$.

Localization and domain-measurability explanation:

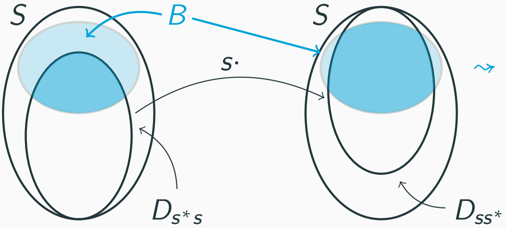


Domain-measurable semigroups

Examples of domain-measurable semigroups:

- All amenable semigroups.
- Non-ame. & domain-measurable: $S = \mathbb{F}_2 \sqcup \{1\}$ and $\mu = \delta_1$.

Localization and domain-measurability explanation:



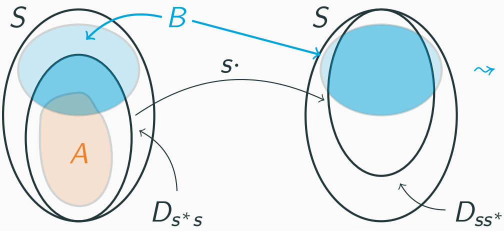
$$\rightsquigarrow \textbf{(1)} \quad \mu(B) = \mu(B \cap D_{s^*s}) = \mu(B \cap D_{ss^*})$$

Domain-measurable semigroups

Examples of domain-measurable semigroups:

- All amenable semigroups.
- Non-ame. & domain-measurable: $S = \mathbb{F}_2 \sqcup \{1\}$ and $\mu = \delta_1$.

Localization and domain-measurability explanation:



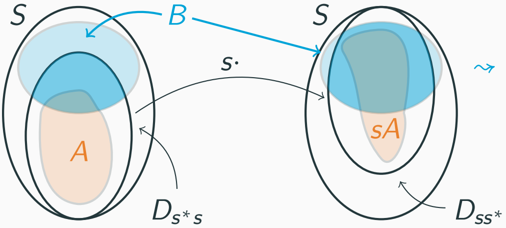
$$\rightsquigarrow \textbf{(1)}\quad \mu(B) = \mu(B \cap D_{s^*s}) = \mu(B \cap D_{ss^*})$$

Domain-measurable semigroups

Examples of domain-measurable semigroups:

- All amenable semigroups.
- Non-ame. & domain-measurable: $S = \mathbb{F}_2 \sqcup \{1\}$ and $\mu = \delta_1$.

Localization and domain-measurability explanation:



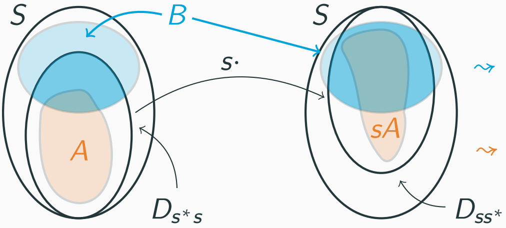
$$\rightsquigarrow \textbf{(1)}\quad \mu(B) = \mu(B \cap D_{s^*s}) = \mu(B \cap D_{ss^*})$$

Domain-measurable semigroups

Examples of domain-measurable semigroups:

- All amenable semigroups.
- Non-ame. & domain-measurable: $S = \mathbb{F}_2 \sqcup \{1\}$ and $\mu = \delta_1$.

Localization and domain-measurability explanation:



$$\rightsquigarrow \textbf{(1)}\ \ \mu(B) = \mu(B \cap D_{s^*s}) = \mu(B \cap D_{ss^*})$$

$$\rightsquigarrow \textbf{(2)}\ \ \mu(A) = \mu(sA)$$

2. Inverse semigroups and Day's amenability

2.2. Paradoxicality and Følner's condition

Paradoxicality and Følner's condition

Følner & paradoxicality $\rightsquigarrow$ from a *domain point of view*

Paradoxicality and Følner's condition

Følner & paradoxicality $\rightsquigarrow$ from a *domain point of view*

Definition (Ara, Lledó, M. - 2020)

Let S be a (countable) inverse semigroup.

(1) S is *domain-Følner* if there are $F_n \Subset S$ such that

$$|s\,(F_n \cap D_{s^*s}) \cup F_n| \,/\, |F_n| \to 1 \text{ for every } s \in S$$

(2) S is *paradoxical* if $\exists s_i, t_j \in S$, $A_i \subset D_{s_i^* s_i}$ and $B_j \subset D_{t_j^* t_j}$ with

$$S = s_1 A_1 \sqcup \cdots \sqcup s_n A_n = t_1 B_1 \sqcup \cdots \sqcup t_m B_m$$
$$\supset A_1 \sqcup \cdots \sqcup A_n \sqcup B_1 \sqcup \cdots \sqcup B_m$$

Paradoxicality and Følner's condition

Følner & paradoxicality $\rightsquigarrow$ from a *domain point of view*

Definition (Ara, Lledó, M. - 2020)

Let S be a (countable) inverse semigroup.

(1) S is *domain-Følner* if there are $F_n \Subset S$ such that

$$|s\,(F_n \cap D_{s^*s}) \cup F_n| \,/\, |F_n| \to 1 \text{ for every } s \in S$$

(2) S is *paradoxical* if $\exists s_i, t_j \in S$, $A_i \subset D_{s_i^* s_i}$ and $B_j \subset D_{t_j^* t_j}$ with

$$\begin{aligned} S &= s_1 A_1 \sqcup \cdots \sqcup s_n A_n = t_1 B_1 \sqcup \cdots \sqcup t_m B_m \\ &\supset A_1 \sqcup \cdots \sqcup A_n \sqcup B_1 \sqcup \cdots \sqcup B_m \end{aligned}$$

Remarks: Note that if $F \cap D_{s^*s} = \varnothing$ then $|s\,(F \cap D_{s^*s}) \cup F| = |F|$

Classical equivalences renovated

Theorem (Ara, Lledó, M. - 2020)

Let S be an inverse semigroup. The following are equivalent:

(1) S is domain-measurable.
(2) S is not paradoxical.
(3) S is domain-Følner.

Classical equivalences renovated

Theorem (Ara, Lledó, M. - 2020)

Let S be an inverse semigroup. The following are equivalent:

(1) S is domain-measurable.
(2) S is not paradoxical.
(3) S is domain-Følner.

(3) $\Rightarrow$ (1): consider $\mu(A) := \lim_{n\to\omega} |A \cap F_n| / |F_n|$. If $B \subset D_{s^*s}$ then

$$|B \cap F_n| = |s(B \cap F_n)| \leq |sB \cap F_n| + |s(F_n \cap D_{s^*s}) \setminus F_n|,$$

and hence, $\mu(B) \leq \mu(sB)$ (the other ineq. follows similarly)

Theorem (Ara, Lledó, M. - 2020)

Let S be an inverse semigroup. The following are equivalent:

(1) S is domain-measurable.
(2) S is not paradoxical.
(3) S is domain-Følner.

(3) $\Rightarrow$ (1): consider $\mu(A) := \lim_{n\to\omega} |A \cap F_n| / |F_n|$. If $B \subset D_{s^*s}$ then

$$|B \cap F_n| = |s(B \cap F_n)| \leq |sB \cap F_n| + |s(F_n \cap D_{s^*s}) \setminus F_n|,$$

and hence, $\mu(B) \leq \mu(sB)$ (the other ineq. follows similarly)

(1) $\Rightarrow$ (2): Par. decompostion + dom. measure $\rightsquigarrow$ Cont. $2 \leq 1$

Classical equivalences renovated

Theorem (Ara, Lledó, M. - 2020)

Let S be an inverse semigroup. The following are equivalent:

(1) S is domain-measurable.
(2) S is not paradoxical.
(3) S is domain-Følner.

(3) $\Rightarrow$ (1): consider $\mu(A) := \lim_{n\to\omega} |A\cap F_n| / |F_n|$. If $B \subset D_{s^*s}$ then

$$|B\cap F_n| = |s(B\cap F_n)| \leq |sB\cap F_n| + |s(F_n \cap D_{s^*s}) \setminus F_n|,$$

and hence, $\mu(B) \leq \mu(sB)$ (the other ineq. follows similarly)

(1) $\Rightarrow$ (2): Par. decompostion + dom. measure $\rightsquigarrow$ Cont. $2 \leq 1$

(2) $\Rightarrow$ (1): based on constructing a *type semigroup* $\mathrm{Typ}(S)$,

i.e., a commutative monoid that encapsulates when

$\mu(A) = \mu(B)$ for $A, B \subset S$

Role of the localization

Recall: Day's amenable $\Leftrightarrow$ localized + dom. measurable

Theorem (Ara, Lledó, M. - 2020)

An inverse semigroup S is amenable iff there are $F_n \Subset S$ such that *eventually* $F_n \subset D_{s^*s}$ and $|sF_n \cup F_n| / |F_n| \to 1$ for all $s \in S$

Role of the localization

Recall: Day's amenable $\Leftrightarrow$ localized + dom. measurable

Theorem (Ara, Lledó, M. - 2020)

An inverse semigroup S is amenable iff there are $F_n \Subset S$ such that *eventually* $F_n \subset D_{s^*s}$ and $|sF_n \cup F_n| / |F_n| \to 1$ for all $s \in S$

Example: the bicyclic monoid $\mathcal{B} = \langle a, a^* \mid a^*a = 1 \rangle$

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \iddots \\
a^3 & a^3a^* & a^3a^{*2} & a^3a^{*3} & \cdots \\
a^2 & a^2a^* & a^2a^{*2} & a^2a^{*3} & \cdots \\
a & aa^* & aa^{*2} & aa^{*3} & \cdots \\
1 & a^* & a^{*2} & a^{*3} & \cdots
\end{array}$$

Role of the localization

Recall: Day's amenable $\Leftrightarrow$ localized + dom. measurable

Theorem (Ara, Lledó, M. - 2020)

An inverse semigroup S is amenable iff there are $F_n \Subset S$ such that *eventually* $F_n \subset D_{s^*s}$ and $|sF_n \cup F_n| / |F_n| \to 1$ for all $s \in S$

Example: the bicyclic monoid $\mathcal{B} = \langle a, a^* \mid a^* a = 1 \rangle$

$\mathcal{B} = D_1$

$$\begin{matrix} \vdots & \vdots & \vdots & \vdots & \iddots \\ a^3 & a^3a^* & a^3a^{*2} & a^3a^{*3} & \cdots \\ a^2 & a^2a^* & a^2a^{*2} & a^2a^{*3} & \cdots \\ a & aa^* & aa^{*2} & aa^{*3} & \cdots \\ 1 & a^* & a^{*2} & a^{*3} & \cdots \end{matrix}$$

Role of the localization

Recall: Day's amenable $\Leftrightarrow$ localized + dom. measurable

Theorem (Ara, Lledó, M. - 2020)

An inverse semigroup S is amenable iff there are $F_n \Subset S$ such that *eventually* $F_n \subset D_{s^*s}$ and $|sF_n \cup F_n| / |F_n| \to 1$ for all $s \in S$

Example: the bicyclic monoid $\mathcal{B} = \langle a, a^* \mid a^* a = 1 \rangle$

D_{aa^*}

$\mathcal{B} = D_1$

$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \iddots \\
a^3 & a^3a^* & a^3a^{*2} & a^3a^{*3} & \dots \\
a^2 & a^2a^* & a^2a^{*2} & a^2a^{*3} & \dots \\
a & aa^* & aa^{*2} & aa^{*3} & \dots \\
\hline
1 & a^* & a^{*2} & a^{*3} & \dots
\end{array}
$$

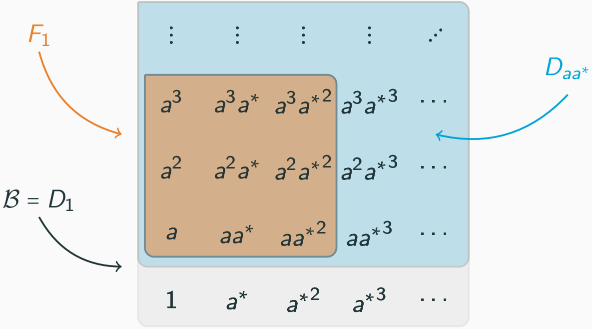
Role of the localization

Recall: Day's amenable $\Leftrightarrow$ localized + dom. measurable

Theorem (Ara, Lledó, M. - 2020)

An inverse semigroup S is amenable iff there are $F_n \Subset S$ such that *eventually* $F_n \subset D_{s^*s}$ and $|sF_n \cup F_n| / |F_n| \to 1$ for all $s \in S$

Example: the bicyclic monoid $\mathcal{B} = \langle a, a^* \mid a^* a = 1 \rangle$



2. Inverse semigroups and Day's amenability

2.3. 2-norm approximations and amenable traces

Left regular representation and the uniform Roe algebra

Goal: use the Følner-type approx. to get approx. of $\mathcal{R}_S$

$\rightsquigarrow$ approximate the traces of $\mathcal{R}_S$ and study them

Left regular representation and the uniform Roe algebra

Goal: use the Følner-type approx. to get approx. of $\mathcal{R}_S$

$\rightsquigarrow$ approximate the traces of $\mathcal{R}_S$ and study them

The *left regular representation* of S is:

$$V\colon S \to \mathcal{B}\left(\ell^2 S\right),\ V_s\delta_x := \begin{cases} \delta_{sx} & \text{if } x \in D_{s^*s} \\ 0 & \text{otherwise} \end{cases}$$

Moreover $\ell^\infty(S) \subset \mathcal{B}\left(\ell^2 S\right)$ as diagonal operators, and

$$\mathcal{R}_S := C^*\left(\{fV_s \mid f \in \ell^\infty(S) \text{ and } s \in S\}\right) \subset \mathcal{B}\left(\ell^2 S\right).$$

Left regular representation and the uniform Roe algebra

Goal: use the Følner-type approx. to get approx. of $\mathcal{R}_S$

$\rightsquigarrow$ approximate the traces of $\mathcal{R}_S$ and study them

The *left regular representation* of S is:

$$V\colon S \to \mathcal{B}\left(\ell^2 S\right), \ V_s\delta_x := \begin{cases} \delta_{sx} & \text{if } x \in D_{s^*s} \\ 0 & \text{otherwise} \end{cases}$$

Moreover $\ell^\infty(S) \subset \mathcal{B}\left(\ell^2 S\right)$ as diagonal operators, and

$$\mathcal{R}_S := C^*\left(\{fV_s \mid f \in \ell^\infty(S) \text{ and } s \in S\}\right) \subset \mathcal{B}\left(\ell^2 S\right).$$

Theorem (classical - for groups!)

For arbitrary groups: $\mathcal{R}_G = \ell^\infty(G) \rtimes_r G = C^*_u(G)$.

Left regular representation and the uniform Roe algebra

Goal: use the Følner-type approx. to get approx. of $\mathcal{R}_S$

$\rightsquigarrow$ approximate the traces of $\mathcal{R}_S$ and study them

The <u>*left regular representation*</u> of S is:

$$V\colon S \to \mathcal{B}\left(\ell^2 S\right), \; V_s\delta_x := \begin{cases} \delta_{sx} & \text{if } x \in D_{s^*s} \\ 0 & \text{otherwise} \end{cases}$$

Moreover $\ell^\infty(S) \subset \mathcal{B}\left(\ell^2 S\right)$ as diagonal operators, and

$$\mathcal{R}_S := C^*\left(\{fV_s \mid f \in \ell^\infty(S) \text{ and } s \in S\}\right) \subset \mathcal{B}\left(\ell^2 S\right).$$

Theorem (classical - for groups!)

For arbitrary groups: $\mathcal{R}_G = \ell^\infty(G) \rtimes_r G = C^*_u(G)$.

Proposition (Lledó, M. - 2021)

One can give $S \curvearrowright \ell^\infty(S)$ such that $\mathcal{R}_S \cong \ell^\infty(S) \rtimes_r S$.

2-norm approximations and amenable traces

Theorem (Ara, Lledó, M. - 2020)

The following (among others) are equivalent:

(1) S is domain-measurable.

(2) S is not paradoxical.

(3) $\mathcal{R}_S$ has an amenable trace.

(4) $\mathcal{R}_S$ is not properly infinite.

2-norm approximations and amenable traces

Theorem (Ara, Lledó, M. - 2020)

The following (among others) are equivalent:

(1) S is domain-measurable.
(2) S is not paradoxical.
(3) $\mathcal{R}_S$ has an amenable trace.
(4) $\mathcal{R}_S$ is not properly infinite.

(1) $\Rightarrow$ (3): the functional $\tau : \mathcal{B}\left(\ell^2 S\right) \xrightarrow{\mathcal{E}} \ell^\infty(S) \xrightarrow{m} \mathbb{C}$, where

- $m(p_B) = \mu(B)$ and extended linearly
- $\mathcal{E} : \mathcal{B}\left(\ell^2 S\right) \to \ell^\infty(S)$, where $a \mapsto \sum_{x \in S} \langle \delta_x, a\delta_x \rangle$

is an amenable trace: $\tau(V_s T) = \tau(T V_s)$ for every $T \in \mathcal{B}\left(\ell^2 S\right)$

2-norm approximations and amenable traces

Theorem (Ara, Lledó, M. - 2020)

The following (among others) are equivalent:

(1) S is domain-measurable.
(2) S is not paradoxical.
(3) $\mathcal{R}_S$ has an amenable trace.
(4) $\mathcal{R}_S$ is not properly infinite.

(1) $\Rightarrow$ (3): the functional $\tau\colon \mathcal{B}\left(\ell^2 S\right) \xrightarrow{\mathcal{E}} \ell^\infty(S) \xrightarrow{m} \mathbb{C}$, where

- $m(p_B) = \mu(B)$ and extended linearly
- $\mathcal{E}\colon \mathcal{B}\left(\ell^2 S\right) \to \ell^\infty(S)$, where $a \mapsto \sum_{x\in S}\langle \delta_x, a\delta_x\rangle$

is an amenable trace: $\tau(V_s T) = \tau(T V_s)$ for every $T \in \mathcal{B}\left(\ell^2 S\right)$

Theorem (Ara, Lledó, M. - 2020)

There is a bijection between the domain-measures of S and the traces of $\mathcal{R}_S$. In particular, all the traces of $\mathcal{R}_S$ are amenable.

3. Coarse geometry of an inverse semigroup

Crash course on coarse geometry

Coarse idea: study metric spaces *globally* instead of *locally*

Example (finite objects seen coarsely)

(X, d) is quasi-isometric to a point $\Leftrightarrow$ $\sup_{x,x' \in X} d(x, x') < \infty$.

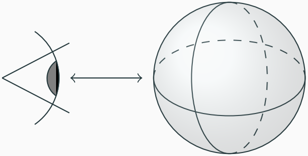
Crash course on coarse geometry

Coarse idea: study metric spaces *globally* instead of *locally*

Example (finite objects seen coarsely)

(X, d) is quasi-isometric to a point $\Leftrightarrow \sup_{x,x' \in X} d(x, x') < \infty$.

Proof:



Crash course on coarse geometry

Coarse idea: study metric spaces *globally* instead of *locally*

Example (finite objects seen coarsely)

(X, d) is quasi-isometric to a point $\Leftrightarrow$ $\sup_{x,x' \in X} d(x, x') < \infty$.

Proof:

Crash course on coarse geometry

Coarse idea: study metric spaces *globally* instead of *locally*

Example (finite objects seen coarsely)

(X, d) is quasi-isometric to a point $\Leftrightarrow$ $\sup_{x,x' \in X} d(x, x') < \infty$.

Proof:

Crash course on coarse geometry

Coarse idea: study metric spaces *globally* instead of *locally*

Example (finite objects seen coarsely)

(X, d) is quasi-isometric to a point $\Leftrightarrow$ $\sup_{x,x' \in X} d(x, x') < \infty$.

Proof:

Crash course on coarse geometry

Coarse idea: study metric spaces *globally* instead of *locally*

Example (finite objects seen coarsely)

(X, d) is quasi-isometric to a point $\Leftrightarrow$ $\sup_{x,x' \in X} d(x, x') < \infty$.

Proof:

Crash course on coarse geometry

Coarse idea: study metric spaces *globally* instead of *locally*

Example (finite objects seen coarsely)

(X, d) is quasi-isometric to a point $\Leftrightarrow$ $\sup_{x,x' \in X} d(x, x') < \infty$.

Proof:

Coarse idea: study metric spaces *globally* instead of *locally*

Example (finite objects seen coarsely)

(X, d) is quasi-isometric to a point $\Leftrightarrow$ $\sup_{x,x' \in X} d(x, x') < \infty$.

Proof:

Group coarse geometry

Recall: Cayley graph construction $\rightsquigarrow G = \langle\, g_1^{\pm 1}, \ldots, g_n^{\pm 1} \,|\, \text{relations}\,\rangle$:

- Graph $\rightsquigarrow \operatorname{Cay}(G, \{g_1, \ldots, g_n\}) := (V, E)$,
- Vertices $\rightsquigarrow V := G$
- Edges $\rightsquigarrow E := \big\{\big(x, g_i^{\pm 1}x\big) \mid x \in G \text{ and } i = 1, \ldots, n\big\}$.

Group coarse geometry

Recall: Cayley graph construction $\rightsquigarrow G = \langle\, g_1^{\pm 1}, \ldots, g_n^{\pm 1} \,|\, \text{relations} \,\rangle$:

- Graph $\rightsquigarrow \mathrm{Cay}\,(G, \{g_1, \ldots, g_n\}) := (V, E)$,
- Vertices $\rightsquigarrow V := G$
- Edges $\rightsquigarrow E := \left\{ \left(x, g_i^{\pm 1} x\right) \mid x \in G \text{ and } i = 1, \ldots, n \right\}$.


$\mathbb{Z} = \langle \pm 2, \pm 3 \rangle$


$\mathbb{Z} = \langle \pm 1 \rangle$

Group coarse geometry

Recall: Cayley graph construction $\rightsquigarrow G = \langle\, g_1^{\pm 1}, \ldots, g_n^{\pm 1} \,|\, \text{relations} \,\rangle$:

- Graph $\rightsquigarrow \mathrm{Cay}\,(G, \{g_1, \ldots, g_n\}) := (V, E)$,
- Vertices $\rightsquigarrow V := G$
- Edges $\rightsquigarrow E := \left\{ \left(x, g_i^{\pm 1} x\right) \mid x \in G \text{ and } i = 1, \ldots, n \right\}$.

 $\mathbb{Z} = \langle \pm 2, \pm 3 \rangle$

 $\mathbb{Z} = \langle \pm 1 \rangle$

Proposition (classical)

The large scale geometry of the Cayley graph of G

does <u>not</u> depend on the generators

Goal: reproduce these constructions for inverse semigroups

i.e., construct Λ_S undirected graph such that $\mathcal{R}_S \cong C_u^*(\Lambda_S)$

Infinite distances, and why they are necessary

Remark: we need to consider *extended* metric spaces

Proposition (Lledó, M. - 2021)

Let $x, y \in S$, and $d\colon S \times S \to [0, \infty]$ be a distance.

(1) If $d(x, y) < \infty$ then $M_{x,y} \in C_u^*(S, d)$
(2) If $x^*x \neq y^*y$ then $M_{x,y} \notin \mathcal{R}_S$

Hence if $x^*x \neq y^*y$ then $d(x, y) = \infty$

Infinite distances, and why they are necessary

Remark: we need to consider *extended* metric spaces

Proposition (Lledó, M. - 2021)

Let $x, y \in S$, and $d\colon S \times S \to [0, \infty]$ be a distance.

(1) If $d(x, y) < \infty$ then $M_{x,y} \in C_u^*(S, d)$
(2) If $x^*x \neq y^*y$ then $M_{x,y} \notin \mathcal{R}_S$

Hence if $x^*x \neq y^*y$ then $d(x, y) = \infty$

Green's relations:

- $x\mathcal{L}y$ if $x^*x = y^*y$
- $x\mathcal{R}y$ if $xx^* = yy^*$
- $\mathcal{H} = \mathcal{L}$ and $\mathcal{R}$
- $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$

Infinite distances, and why they are necessary

Remark: we need to consider *extended* metric spaces

Proposition (Lledó, M. - 2021)

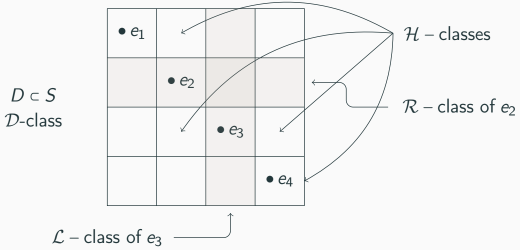
Let $x, y \in S$, and $d\colon S \times S \to [0, \infty]$ be a distance.

(1) If $d(x, y) < \infty$ then $M_{x,y} \in C^*_u(S, d)$
(2) If $x^*x \neq y^*y$ then $M_{x,y} \notin \mathcal{R}_S$

Hence if $x^*x \neq y^*y$ then $d(x, y) = \infty$

Green's relations:

- $x\mathcal{L}y$ if $x^*x = y^*y$
- $x\mathcal{R}y$ if $xx^* = yy^*$
- $\mathcal{H} = \mathcal{L}$ and $\mathcal{R}$
- $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$



Schützenberger graphs I: $\mathcal{L}$-classes

Definition (Schützenberger - 1959)

Let $S = \langle K \rangle$, where $K = K^*$. Given an $\mathcal{L}$-class $L \subset S$, let Λ_L be

- the graph whose vertices are the points of L and
- where $x, y \in L$ are joined by a k-labeled edge if $kx = y$.

Likewise, let $\Lambda_S = \sqcup_{e \in E(S)} \Lambda_{L_e}$.

Schützenberger graphs I: $\mathcal{L}$-classes

Definition (Schützenberger - 1959)

Let $S = \langle K \rangle$, where $K = K^*$. Given an $\mathcal{L}$-class $L \subset S$, let Λ_L be

- the graph whose vertices are the points of L and
- where $x, y \in L$ are joined by a k-labeled edge if $kx = y$.

Likewise, let $\Lambda_S = \sqcup_{e \in E(S)} \Lambda_{L_e}$.

"Algorithm":

• Construct the (left)
Cayley graph w.r.t. K

• Erase directed arrows

Schützenberger graphs I: $\mathcal{L}$-classes

Definition (Schützenberger - 1959)

Let $S = \langle K \rangle$, where $K = K^*$. Given an $\mathcal{L}$-class $L \subset S$, let Λ_L be

- the graph whose vertices are the points of L and
- where $x, y \in L$ are joined by a k-labeled edge if $kx = y$.

Likewise, let $\Lambda_S = \sqcup_{e \in E(S)} \Lambda_{L_e}$.

Example: $\mathcal{B} := \langle a, a^* \mid a^* a = 1 \rangle$

"Algorithm":

• Construct the (left)

Cayley graph w.r.t. K

• Erase directed arrows

Schützenberger graphs I: $\mathcal{L}$-classes

Definition (Schützenberger - 1959)

Let $S = \langle K \rangle$, where $K = K^*$. Given an $\mathcal{L}$-class $L \subset S$, let Λ_L be

- the graph whose vertices are the points of L and
- where $x, y \in L$ are joined by a k-labeled edge if $kx = y$.

Likewise, let $\Lambda_S = \sqcup_{e \in E(S)} \Lambda_{L_e}$.

Example: $\mathcal{B} := \langle a, a^* \mid a^* a = 1 \rangle$

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \iddots \\
a^3 & a^3 a^* & a^3 a^{*2} & a^3 a^{*3} & \dots \\
| & | & | & | & \\
a^2 & a^2 a^* & a^2 a^{*2} & a^2 a^{*3} & \dots \\
| & | & | & | & \\
a & aa^* & aa^{*2} & aa^{*3} & \dots \\
| & | & | & | & \\
1 \longrightarrow & a^* \longrightarrow & a^{*2} \longrightarrow & a^{*3} & \dots
\end{array}$$

"Algorithm":

- Construct the (left) Cayley graph w.r.t. K
- Erase directed arrows

Schützenberger graphs I: $\mathcal{L}$-classes

Definition (Schützenberger - 1959)

Let $S = \langle K \rangle$, where $K = K^*$. Given an $\mathcal{L}$-class $L \subset S$, let Λ_L be

- the graph whose vertices are the points of L and
- where $x, y \in L$ are joined by a k-labeled edge if $kx = y$.

Likewise, let $\Lambda_S = \sqcup_{e \in E(S)} \Lambda_{L_e}$.

"Algorithm":

- Construct the (left) Cayley graph w.r.t. K
- Erase directed arrows

Example: $\mathcal{B} := \langle a, a^* \mid a^* a = 1 \rangle$

$$\begin{tikzcd}[row sep=small]
\vdots & \vdots & \vdots & \vdots & \iddots \\
a^3 & a^3a^* & a^3a^{*2} & a^3a^{*3} & \ldots \\
a^2 \arrow[u, dash] & a^2a^* \arrow[u, dash] & a^2a^{*2} \arrow[u, dash] & a^2a^{*3} \arrow[u, dash] & \ldots \\
a \arrow[u, dash] & aa^* \arrow[u, dash] & aa^{*2} \arrow[u, dash] & aa^{*3} \arrow[u, dash] & \ldots \\
1 \arrow[u, dash] \arrow[r] & a^* \arrow[u, dash] \arrow[r] & a^{*2} \arrow[u, dash] \arrow[r] & a^{*3} \arrow[u, dash] & \ldots
\end{tikzcd}$$

Definition (Schützenberger - 1959)

Let $S = \langle K \rangle$, where $K = K^*$. Given an $\mathcal{L}$-class $L \subset S$, let Λ_L be

- the graph whose vertices are the points of L and
- where $x, y \in L$ are joined by a k-labeled edge if $kx = y$.

Likewise, let $\Lambda_S = \sqcup_{e \in E(S)} \Lambda_{L_e}$.

Example: $\mathcal{B} := \langle a, a^* \mid a^* a = 1 \rangle$

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \iddots \\
a^3 \arrow[d, dash] & a^3a^* \arrow[d, dash] & a^3a^{*2} \arrow[d, dash] & a^3a^{*3} \arrow[d, dash] & \dots \\
a^2 \arrow[d, dash] & a^2a^* \arrow[d, dash] & a^2a^{*2} \arrow[d, dash] & a^2a^{*3} \arrow[d, dash] & \dots \\
a \arrow[d, dash] & aa^* \arrow[d, dash] & aa^{*2} \arrow[d, dash] & aa^{*3} \arrow[d, dash] & \dots \\
1 \arrow[r] & a^* \arrow[r] & a^{*2} \arrow[r] & a^{*3} & \dots
\end{tikzcd}$$

"Algorithm":

- Construct the (left) Cayley graph w.r.t. K
- Erase directed arrows

Schützenberger graphs I: $\mathcal{L}$-classes

Definition (Schützenberger - 1959)

Let $S = \langle K \rangle$, where $K = K^*$. Given an $\mathcal{L}$-class $L \subset S$, let Λ_L be

- the graph whose vertices are the points of L and
- where $x, y \in L$ are joined by a k-labeled edge if $kx = y$.

Likewise, let $\Lambda_S = \sqcup_{e \in E(S)} \Lambda_{L_e}$.

Example: $\mathcal{B} := \langle a, a^* \mid a^* a = 1 \rangle$

$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \iddots \\
a^3 & a^3a^* & a^3a^{*2} & a^3a^{*3} & \dots \\
| & | & | & | & \\
a^2 & a^2a^* & a^2a^{*2} & a^2a^{*3} & \dots \\
| & | & | & | & \\
a & aa^* & aa^{*2} & aa^{*3} & \dots \\
| & | & | & | & \\
1 & a^* & a^{*2} & a^{*3} & \dots
\end{array}
$$

"Algorithm":

- Construct the (left) Cayley graph w.r.t. K
- Erase directed arrows

Schützenberger graphs I: $\mathcal{L}$-classes

Definition (Schützenberger - 1959)

Let $S = \langle K \rangle$, where $K = K^*$. Given an $\mathcal{L}$-class $L \subset S$, let Λ_L be

- the graph whose vertices are the points of L and
- where $x, y \in L$ are joined by a k-labeled edge if $kx = y$.

Likewise, let $\Lambda_S = \sqcup_{e \in E(S)} \Lambda_{L_e}$.

Example: $\mathcal{B} := \langle a, a^* \mid a^* a = 1 \rangle$

"Algorithm":

- Construct the (left) Cayley graph w.r.t. K
- Erase directed arrows

$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \iddots \\
a^3 & a^3 a^* & a^3 a^{*2} & a^3 a^{*3} & \ldots \\
| & | & | & | & \\
a^2 & a^2 a^* & a^2 a^{*2} & a^2 a^{*3} & \ldots \\
| & | & | & | & \\
a & aa^* & aa^{*2} & aa^{*3} & \ldots \\
| & | & | & | & \\
1 & a^* & a^{*2} & a^{*3} & \ldots
\end{array}
$$

Schützenberger graphs II: right invariance

Remark: not all graphs are Cayley graphs. However:

Theorem (Stephen - 1990)

$$\{\text{connected graphs}\} = \{\text{Schützenberger graphs}\}$$

Schützenberger graphs II: right invariance

Remark: not all graphs are Cayley graphs. However:

Theorem (Stephen - 1990)

$$\{\text{connected graphs}\} = \{\text{Schützenberger graphs}\}$$

Lemma (right invariance)

Let $S = \langle K \rangle$. If $x \in D_{s^*s}$ then $d(x, sx) \leq d(s^*s, s)$.

In particular, if $xx^* = s^*s$ then $d(x, sx) = d(s^*s, s)$.

Proof/consequence:

L_{s^*s}

Schützenberger graphs II: right invariance

Remark: not all graphs are Cayley graphs. However:

Theorem (Stephen - 1990)

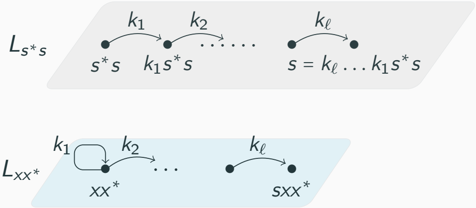
$$\{\text{connected graphs}\} = \{\text{Schützenberger graphs}\}$$

Lemma (right invariance)

Let $S = \langle K \rangle$. If $x \in D_{s^*s}$ then $d(x, sx) \leq d(s^*s, s)$.

In particular, if $xx^* = s^*s$ then $d(x, sx) = d(s^*s, s)$.

Proof/consequence:



Schützenberger graphs II: right invariance

Remark: not all graphs are Cayley graphs. However:

Theorem (Stephen - 1990)

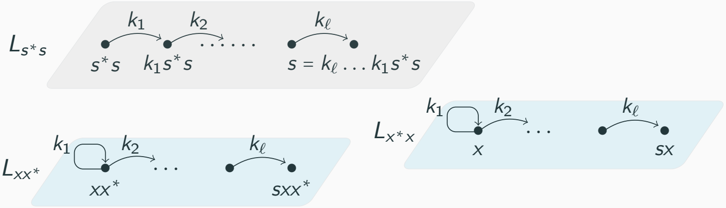
$$\{\text{connected graphs}\} = \{\text{Schützenberger graphs}\}$$

Lemma (right invariance)

Let $S = \langle K \rangle$. If $x \in D_{s^*s}$ then $d(x, sx) \leq d(s^*s, s)$.

In particular, if $xx^* = s^*s$ then $d(x, sx) = d(s^*s, s)$.

Proof/consequence:



Schützenberger graphs II: right invariance

Remark: not all graphs are Cayley graphs. However:

Theorem (Stephen - 1990)

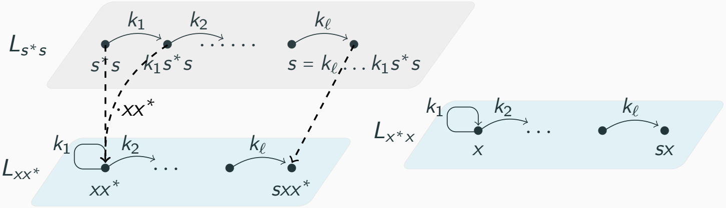
$$\{\text{connected graphs}\} = \{\text{Schützenberger graphs}\}$$

Lemma (right invariance)

Let $S = \langle K \rangle$. If $x \in D_{s^*s}$ then $d(x, sx) \leq d(s^*s, s)$.

In particular, if $xx^* = s^*s$ then $d(x, sx) = d(s^*s, s)$.

Proof/consequence:



Schützenberger graphs II: right invariance

Remark: not all graphs are Cayley graphs. However:

Theorem (Stephen - 1990)

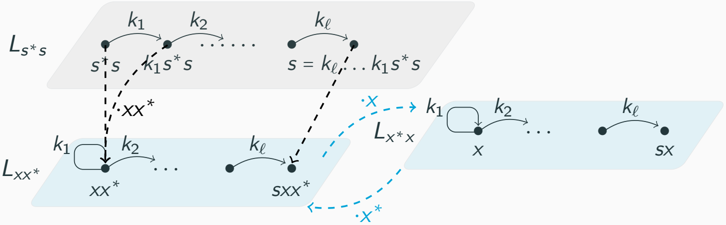
$$\{\text{connected graphs}\} = \{\text{Schützenberger graphs}\}$$

Lemma (right invariance)

Let $S = \langle K \rangle$. If $x \in D_{s^*s}$ then $d(x, sx) \leq d(s^*s, s)$.

In particular, if $xx^* = s^*s$ then $d(x, sx) = d(s^*s, s)$.

Proof/consequence:



3. Coarse geometry of an inverse semigroup

3.1. Finite labelings and uniform Roe algebras

Finite labeling I: $\mathcal{R}_S$ as a uniform Roe algebra

Proposition (Lledó, M. - 2021)

Let $S = \langle K \rangle$. Then $\mathcal{R}_S \subset C_u^* (\Lambda_S)$.

Proof: generators of $\mathcal{R}_S$ have finite propagation.

Finite labeling I: $\mathcal{R}_S$ as a uniform Roe algebra

Proposition (Lledó, M. - 2021)

Let $S = \langle K \rangle$. Then $\mathcal{R}_S \subset C_u^*(\Lambda_S)$.

Proof: generators of $\mathcal{R}_S$ have finite propagation.

Warning: Sometimes $\mathcal{R}_S \subsetneq C_u^*(\Lambda_S)$, e.g., $S = (\mathbb{N}, \min)$ where

$$\mathcal{R}_S = c_0(\mathbb{N}) \subsetneq \ell^\infty(\mathbb{N}) = C_u^*(\Lambda_S)$$

Finite labeling I: $\mathcal{R}_S$ as a uniform Roe algebra

Proposition (Lledó, M. - 2021)

Let $S = \langle K \rangle$. Then $\mathcal{R}_S \subset C_u^*(\Lambda_S)$.

Proof: generators of $\mathcal{R}_S$ have finite propagation.

Warning: Sometimes $\mathcal{R}_S \subsetneq C_u^*(\Lambda_S)$, e.g., $S = (\mathbb{N}, \min)$ where

$$\mathcal{R}_S = c_0(\mathbb{N}) \subsetneq \ell^\infty(\mathbb{N}) = C_u^*(\Lambda_S)$$

Key: for large $r \geq 0$ $\nexists$ $F \Subset S$ labeling the paths in Λ_S of length r

Finite labeling I: $\mathcal{R}_S$ as a uniform Roe algebra

Proposition (Lledó, M. - 2021)

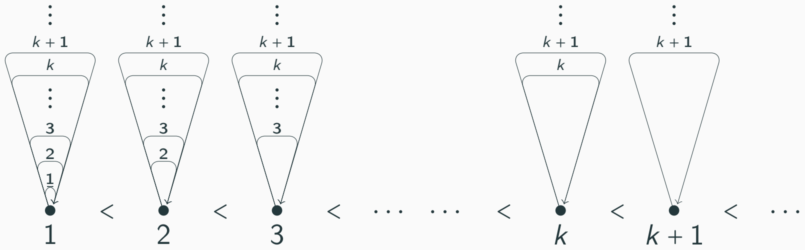
Let $S = \langle K \rangle$. Then $\mathcal{R}_S \subset C_u^*(\Lambda_S)$.

Proof: generators of $\mathcal{R}_S$ have finite propagation.

Warning: Sometimes $\mathcal{R}_S \subsetneq C_u^*(\Lambda_S)$, e.g., $S = (\mathbb{N}, \min)$ where

$$\mathcal{R}_S = c_0(\mathbb{N}) \subsetneq \ell^\infty(\mathbb{N}) = C_u^*(\Lambda_S)$$

Key: for large $r \geq 0 \nexists\ F \Subset S$ labeling the paths in Λ_S of length r



Finite labeling I: $\mathcal{R}_S$ as a uniform Roe algebra

Proposition (Lledó, M. - 2021)

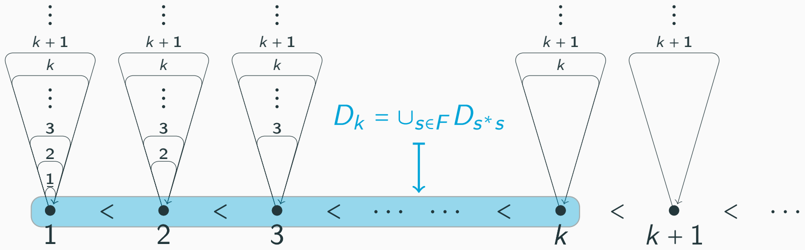
Let $S = \langle K \rangle$. Then $\mathcal{R}_S \subset C_u^*(\Lambda_S)$.

Proof: generators of $\mathcal{R}_S$ have finite propagation.

Warning: Sometimes $\mathcal{R}_S \subsetneq C_u^*(\Lambda_S)$, e.g., $S = (\mathbb{N}, \min)$ where

$$\mathcal{R}_S = c_0(\mathbb{N}) \subsetneq \ell^\infty(\mathbb{N}) = C_u^*(\Lambda_S)$$

Key: for large $r \geq 0$ $\nexists$ $F \Subset S$ labeling the paths in Λ_S of length r



Finite labeling II: a picture to *top* the explanation

Definition (Lledó, M. - 2021)

Let $S = \langle K \rangle$. We say (S, K) admits a <u>*finite labeling*</u> if

for any $r \geq 0$ there is $F \Subset S$ such that if $d(s^*s, s) \leq r$

then there is $m \in F$ such that $ms^*s = s$.

Finite labeling II: a picture to *top* the explanation

Definition (Lledó, M. - 2021)

Let $S = \langle K \rangle$. We say (S, K) admits a <u>*finite labeling*</u> if

$$\text{for any } r \geq 0 \text{ there is } F \Subset S \text{ such that if } d(s^*s, s) \leq r$$
$$\text{then there is } m \in F \text{ such that } ms^*s = s.$$

Examples:

(1) Finitely generated semigroups
(2) F-inverse semigroups with Λ_S of bounded geometry

Finite labeling II: a picture to *top* the explanation

Definition (Lledó, M. - 2021)

Let $S = \langle K \rangle$. We say (S, K) admits a *finite labeling* if

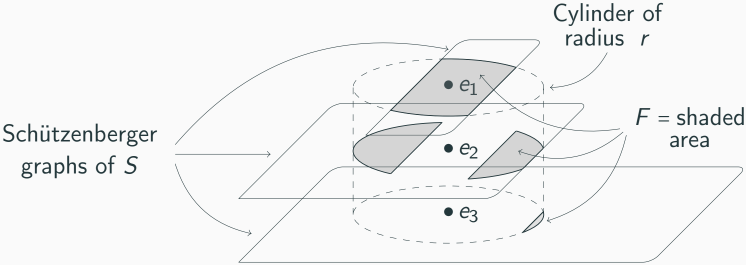
for any $r \geq 0$ there is $F \Subset S$ such that if $d(s^*s, s) \leq r$

then there is $m \in F$ such that $ms^*s = s$.

Examples:

(1) Finitely generated semigroups

(2) F-inverse semigroups with Λ_S of bounded geometry



$\vdots$

Finite labeling III: importance

Theorem (Lledó, M. - 2021)

(S, K) admits a finite labeling $\Leftrightarrow \mathcal{R}_S = \ell^\infty(S) \rtimes_r S = C_u^*(\Lambda_S)$.

Proof $\Rightarrow$: take $T \in C_u^*(\Lambda_S)$ such that

$$\operatorname{prop}(T) = \sup\{d(x, y) \mid \langle \delta_y, T\delta_x \rangle \neq 0\} = r < \infty$$

Choose $F \Subset S$ labeling all the r-paths in Λ_S $\rightsquigarrow$ for all $x, y \in S$ with $d(x, y) < r$ there is $t_{x,y} \in F$ such that $t_{x,y}x = y$. Then

Finite labeling III: importance

Theorem (Lledó, M. - 2021)

(S, K) admits a finite labeling $\Leftrightarrow \mathcal{R}_S = \ell^\infty(S) \rtimes_r S = C_u^*(\Lambda_S)$.

Proof $\Rightarrow$: take $T \in C_u^*(\Lambda_S)$ such that

$$\mathrm{prop}(T) = \sup\{d(x,y) \mid \langle \delta_y, T\delta_x \rangle \neq 0\} = r < \infty$$

Choose $F \Subset S$ labeling all the r-paths in Λ_S $\rightsquigarrow$ for all $x, y \in S$ with $d(x,y) < r$ there is $t_{x,y} \in F$ such that $t_{x,y}x = y$. Then

$$\mathcal{R}_S \ni \sum_{s \in F} \xi_s V_s = T, \text{ where } \xi_s(y) = \begin{cases} \langle \delta_y, T\delta_{s^*y} \rangle & \text{if } s = t_{s^*y,y} \\ 0 & \text{otherwise} \end{cases}$$

Finite labeling III: importance

Theorem (Lledó, M. - 2021)

(S, K) admits a finite labeling $\Leftrightarrow$ $\mathcal{R}_S = \ell^\infty(S) \rtimes_r S = C_u^*(\Lambda_S)$.

Proof $\Rightarrow$: take $T \in C_u^*(\Lambda_S)$ such that

$$\mathrm{prop}(T) = \sup\{d(x,y) \mid \langle \delta_y, T\delta_x \rangle \neq 0\} = r < \infty$$

Choose $F \Subset S$ labeling all the r-paths in Λ_S $\rightsquigarrow$ for all $x, y \in S$ with $d(x,y) < r$ there is $t_{x,y} \in F$ such that $t_{x,y}x = y$. Then

$$\mathcal{R}_S \ni \sum_{s \in F} \xi_s V_s = T, \quad \text{where } \xi_s(y) = \begin{cases} \langle \delta_y, T\delta_{s^* y} \rangle & \text{if } s = t_{s^* y, y} \\ 0 & \text{otherwise} \end{cases}$$

Remark: somehow, the Theorem says that if (S, K) is <u>not</u> FL then maybe change your generating set K...

Theorem (classical - for groups!)

For arbitrary groups: $\mathcal{R}_G = \ell^\infty(G) \rtimes_r G = C_u^*(G)$.

3. Coarse geometry of an inverse semigroup

3.2. Domain-measurability as a coarse invariant

Domain-measurability in the Schützenberger graphs

Følner condition: $s\,(F_n \cap D_{s^*s}) \rightsquigarrow$ only moving in $\mathcal{L}$-classes.

Domain-measurability in the Schützenberger graphs

Følner condition: $s\,(F_n \cap D_{s^*s}) \rightsquigarrow$ only moving in $\mathcal{L}$-classes.

Therefore:

Domain-measurable $\Leftrightarrow$ Schützenberger graphs have *small growth*:

Proposition (Lledó, M. - 2021)

Suppose $S = \langle K \rangle$ admits a FL. S is domain-measurable iff
for every $r, \varepsilon > 0$ there is $F \Subset S$ such that $|\mathcal{N}_r^+ F| \leq (1+\varepsilon)\,|F|$.

Domain-measurability in the Schützenberger graphs

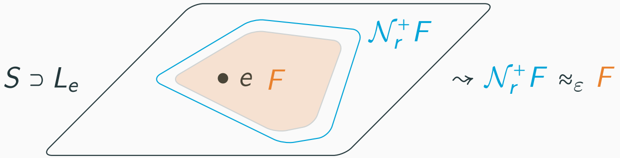
Følner condition: $s\,(F_n \cap D_{s^*s}) \rightsquigarrow$ only moving in $\mathcal{L}$-classes.

Therefore:

Domain-measurable $\Leftrightarrow$ Schützenberger graphs have *small growth*:

Proposition (Lledó, M. - 2021)

Suppose $S = \langle K \rangle$ admits a FL. S is domain-measurable iff
for every $r, \varepsilon > 0$ there is $F \Subset S$ such that $|\mathcal{N}_r^+ F| \leq (1+\varepsilon)\,|F|$.



Quasi-isometric invariance of domain-measurability

Theorem (Lledó, M. - 2021)

Let S and T be quasi-isometric and admitting finite labelings.

If T is domain-measurable then so is S.

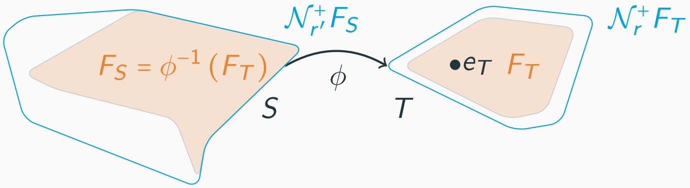
Quasi-isometric invariance of domain-measurability

Theorem (Lledó, M. - 2021)

Let S and T be quasi-isometric and admitting finite labelings.

If T is domain-measurable then so is S.

Proof:



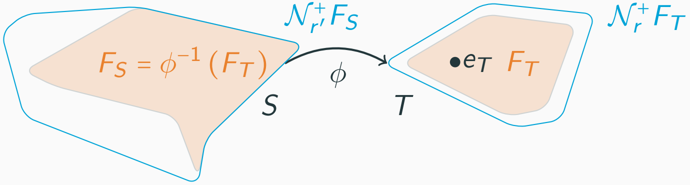
Quasi-isometric invariance of domain-measurability

Theorem (Lledó, M. - 2021)

Let S and T be quasi-isometric and admitting finite labelings.

If T is domain-measurable then so is S.

Proof:



Remark: amenability is <u>not</u> a q.i. invariant:

- $S = \mathbb{F}_2 \sqcup \{0\}$ and $T = \mathbb{F}_2 \sqcup \{1\}$
- $\phi\colon S \to T$, where $\phi(\omega) = \omega$ and $\phi(0) = 1$ is a q.i.

3. Coarse geometry of an inverse semigroup

3.3. Property A as a C*-property

Schützenberger graphs and property A

Definition (Yu - 1999)

(X, d) has *property A* if for every $r, \varepsilon > 0$ there is
$\xi : X \to \ell^1(X)_1^+$ and $c > 0$ such that $\operatorname{supp}(\xi_x) \subset B_c(x)$ and
$\|\xi_x - \xi_y\|_1 \leq \varepsilon$ for every $x, y \in X$ such that $d(x, y) \leq r$.

Schützenberger graphs and property A

Definition (Yu - 1999)

(X,d) has *property A* if for every $r, \varepsilon > 0$ there is

$$\xi\colon X \to \ell^1(X)_1^+ \text{ and } c > 0 \text{ such that } \operatorname{supp}(\xi_x) \subset B_c(x) \text{ and}$$

$$\|\xi_x - \xi_y\|_1 \leq \varepsilon \text{ for every } x, y \in X \text{ such that } d(x,y) \leq r.$$

Remarks:

- Property A generalizes amenability for groups (**not** in general)
- Non-property A groups are hard to come by

Definition (Yu - 1999)

(X,d) has *property A* if for every $r,\varepsilon > 0$ there is

$$\xi\colon X \to \ell^1(X)_1^+ \text{ and } c > 0 \text{ such that } \operatorname{supp}(\xi_x) \subset B_c(x) \text{ and}$$
$$\|\xi_x - \xi_y\|_1 \leq \varepsilon \text{ for every } x,y \in X \text{ such that } d(x,y) \leq r.$$

Remarks:

- Property A generalizes amenability for groups (**not** in general)
- Non-property A groups are hard to come by

Theorem (Ozawa - 2000)

Let G be a countable group. The following are equivalent:

(1) G has property A.
(2) $\mathcal{R}_G$ is a nuclear C*-algebra.
(3) $C_r^*(G)$ is an exact C*-algebra.

Property A, nuclearity and exactness for inverse semigroups

Theorem (Lledó, M. - 2021)

Let $S = \langle K \rangle$. Consider:

(1) Λ_S has property A (as a graph).
(2) $\mathcal{R}_S$ is a nuclear C*-algebra.
(3) $C_r^*(S)$ is an exact C*-algebra.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3), and (3) $\Rightarrow$ (1) when (S, K) admits a FL.

Property A, nuclearity and exactness for inverse semigroups

Theorem (Lledó, M. - 2021)

Let $S = \langle K \rangle$. Consider:

(1) Λ_S has property A (as a graph).
(2) $\mathcal{R}_S$ is a nuclear C*-algebra.
(3) $C_r^*(S)$ is an exact C*-algebra.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3), and (3) $\Rightarrow$ (1) when (S, K) admits a FL.

Proof:

- Usual arguments for (1) $\Rightarrow$ (2) $\Rightarrow$ (3)
- Nuclearity = exactness for $\mathcal{R}_S = C_u^*(\Lambda_S)$ [Sako 2021]

Theorem (Lledó, M. - 2021)

Let $S = \langle K \rangle$. Consider:

(1) Λ_S has property A (as a graph).
(2) $\mathcal{R}_S$ is a nuclear C*-algebra.
(3) $C_r^*(S)$ is an exact C*-algebra.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3), and (3) $\Rightarrow$ (1) when (S, K) admits a FL.

Proof:

- Usual arguments for (1) $\Rightarrow$ (2) $\Rightarrow$ (3)
- Nuclearity = exactness for $\mathcal{R}_S = C_u^*(\Lambda_S)$ [Sako 2021]

Remark: (1) $\Rightarrow$ (2) $\rightsquigarrow$ nuclearity does not pass to sub-algebras!

4. The universal groupoid and its amenability

Paterson's universal groupoid and its amenability

Theorem (Paterson - 1999)

Given S there is $G_{\mathcal{U}}(S) = \{\text{filters of } E(S)\} \rtimes_\theta S$ such that

$$C_r^*(S) = C_r^*(G_{\mathcal{U}}(S)) \text{ and } C^*(S) = C^*(G_{\mathcal{U}}(S))$$

Paterson's universal groupoid and its amenability

Theorem (Paterson - 1999)

Given S there is $G_{\mathcal{U}}(S) = \{\text{filters of } E(S)\} \rtimes_\theta S$ such that

$$C_r^*(S) = C_r^*(G_{\mathcal{U}}(S)) \text{ and } C^*(S) = C^*(G_{\mathcal{U}}(S))$$

Recall: for arbitrary (discrete) groups:

$$G \text{ is amenable} \Leftrightarrow C_r^*(G) \text{ is nuclear} \Leftrightarrow C_r^*(G) = C^*(G)$$

And this has become a driving force of groupoid amenability:

Theorem (Anantharaman-Delaroche and Renault)

An (étale) groupoid $\mathcal{G}$ is amenable $\Leftrightarrow$ $C_r^*(\mathcal{G})$ is nuclear.

Paterson's universal groupoid and its amenability

Theorem (Paterson - 1999)

Given S there is $G_{\mathcal{U}}(S) = \{\text{filters of } E(S)\} \rtimes_\theta S$ such that

$$C_r^*(S) = C_r^*(G_{\mathcal{U}}(S)) \text{ and } C^*(S) = C^*(G_{\mathcal{U}}(S))$$

Recall: for arbitrary (discrete) groups:

$$G \text{ is amenable} \Leftrightarrow C_r^*(G) \text{ is nuclear} \Leftrightarrow C_r^*(G) = C^*(G)$$

And this has become a driving force of groupoid amenability:

Theorem (Anantharaman-Delaroche and Renault)

An (étale) groupoid $\mathcal{G}$ is amenable $\Leftrightarrow$ $C_r^*(\mathcal{G})$ is nuclear.

Question:

- Relation between $G_{\mathcal{U}}(S)$ amenable and S amenable?
- How does $G_{\mathcal{U}}(S)$ enter the picture?

Groupoid amenability vs domain-measurability

Question: relation between $G_{\mathcal{U}}(S)$ amenable and S amenable?

Groupoid amenability vs domain-measurability

Question: relation between $G_{\mathcal{U}}(S)$ amenable and S amenable?

Short answer: no

Groupoid amenability vs domain-measurability

Question: relation between $G_{\mathcal{U}}(S)$ amenable and S amenable?

Short answer: no relation, as

(1) S amenable/domain-measurable $\rightsquigarrow$ traces in $\mathcal{R}_S$
(2) $G_{\mathcal{U}}(S)$ amenable $\rightsquigarrow$ nuclearity of $C_r^*(S)$ $\rightsquigarrow$ nuclearity of $\mathcal{R}_S$

Groupoid amenability vs domain-measurability

Question: relation between $G_{\mathcal{U}}(S)$ amenable and S amenable?

Short answer: no relation, as

(1) S amenable/domain-measurable $\rightsquigarrow$ traces in $\mathcal{R}_S$
(2) $G_{\mathcal{U}}(S)$ amenable $\rightsquigarrow$ nuclearity of $C_r^*(S) \rightsquigarrow$ nuclearity of $\mathcal{R}_S$

Examples:

(i) $\mathbb{F}_2 \sqcup \{0\}$ is amenable but non-nuclear reduced C*-algebra
(Can also be done with a *box space* with a prop. (T) group)
(ii) $C_r^*(\mathbb{F}_2 \sqcup \{0\}) \neq C^*(\mathbb{F}_2 \sqcup \{0\})$
(iii) Nica gave a non-amenable semigroup with nuclear C*-algebra

Groupoid amenability and property A

However, property A of $\Lambda_S \rightsquigarrow$ exactness of $C_r^*(S)$:

Proposition (Lledó, M. - 2021)

Let $S = \langle K \rangle$ with (S, K) admitting a finite labeling.

If $G_{\mathcal{U}}(S)$ is amenable then Λ_S has property A.

Groupoid amenability and property A

However, property A of $\Lambda_S \rightsquigarrow$ exactness of $C_r^*(S)$:

Proposition (Lledó, M. - 2021)

Let $S = \langle K \rangle$ with (S, K) admitting a finite labeling.

If $G_{\mathcal{U}}(S)$ is amenable then Λ_S has property A.

Proof: $G_{\mathcal{U}}(S)$ amenable $\Leftrightarrow C_r^*(S) = C_r^*(G_{\mathcal{U}}(S))$ nuclear

$$\Rightarrow C_r^*(S) \text{ exact} \Leftrightarrow \Lambda_S \text{ has prop. A}$$

Groupoid amenability and property A

However, property A of $\Lambda_S \rightsquigarrow$ exactness of $C^*_r(S)$:

Proposition (Lledó, M. - 2021)

Let $S = \langle K \rangle$ with (S, K) admitting a finite labeling.

If $G_{\mathcal{U}}(S)$ is amenable then Λ_S has property A.

Proof: $G_{\mathcal{U}}(S)$ amenable $\Leftrightarrow C^*_r(S) = C^*_r(G_{\mathcal{U}}(S))$ nuclear

$\Rightarrow C^*_r(S)$ exact $\Leftrightarrow \Lambda_S$ has prop. A

Remarks:

- Alternate argument via the definitions involved $\rightsquigarrow$ not C*
- Is <u>not</u> an equivalence, as $\mathbb{F}_2$ has prop. A and is not amenable

5. Norm approximations and quasi-diagonality

From QD to domain-measurability

Definition/Theorem (Voiculescu, based on Halmos)

A C*-algebra A is *quasi-diagonal* if there are u.c.p. maps $\varphi_n \colon A \to M_{k(n)}$ such that for all $a, b \in A$

$$\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0 \quad \text{and} \quad \|\varphi_n(a)\| \to \|a\|$$

From QD to domain-measurability

Definition/Theorem (Voiculescu, based on Halmos)

A C*-algebra A is *quasi-diagonal* if there are u.c.p. maps $\varphi_n \colon A \to M_{k(n)}$ such that for all $a, b \in A$

$$\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0 \quad \text{and} \quad \|\varphi_n(a)\| \to \|a\|$$

Theorem (Ara, Lledó, M. - 2020)

If $C^*_r(S)$ is quasi-diagonal then S is domain-measurable. The reverse implication is false.

Sketch of proof:

As $\|\cdot\|_2 \leq \|\cdot\| \rightsquigarrow$ QD is stronger than the previous 2-approx.
$\rightsquigarrow$ dom. measurable = amenable trace = 2-norm approx.

(Any wk-* cluster point of $\left\{\mathrm{tr}_{k(n)} \circ \varphi_n\right\}_{n\in\mathbb{N}}$ gives a dom. measure)

From QD to domain-measurability

Definition/Theorem (Voiculescu, based on Halmos)

A C*-algebra A is *quasi-diagonal* if there are u.c.p. maps $\varphi_n \colon A \to M_{k(n)}$ such that for all $a, b \in A$

$$\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0 \quad \text{and} \quad \|\varphi_n(a)\| \to \|a\|$$

Theorem (Ara, Lledó, M. - 2020)

If $C_r^*(S)$ is quasi-diagonal then S is domain-measurable. The reverse implication is false.

Sketch of proof:

As $\|\cdot\|_2 \leq \|\cdot\| \rightsquigarrow$ QD is stronger than the previous 2-approx.
$\rightsquigarrow$ dom. measurable = amenable trace = 2-norm approx.

(Any wk-* cluster point of $\left\{\mathrm{tr}_{k(n)} \circ \varphi_n\right\}_{n\in\mathbb{N}}$ gives a dom. measure)

Lastly, $\mathcal{B} = \langle a, a^* \mid a^* a = 1\rangle$ is amenable but $V_{aa^*} \sim 1$ in $C_r^*(\mathcal{B})$

Groupoid-TWW approach to going back

Goal: use algebraic properties of $S \rightsquigarrow$ prove quasi-diagonality.

There currently are two strategies:

Groupoid-TWW approach to going back

Goal: use algebraic properties of $S \rightsquigarrow$ prove quasi-diagonality.

There currently are two strategies:

(TWW) Nuclearity + UCT + faithful trace $\Rightarrow C_r^*(S)$ is QD

(Alt) Exploit the inner structure of S

Groupoid-TWW approach to going back

Goal: use algebraic properties of $S \rightsquigarrow$ prove quasi-diagonality.
There currently are two strategies:

(TWW) Nuclearity + UCT + faithful trace $\Rightarrow C_r^*(S)$ is QD
(Alt) Exploit the inner structure of S

Unfortunately, (TWW) does not quite work in this context:

Proposition (M. - 2021)

Suppose $G_{\mathcal{U}}(S)$ is minimal and $C_r^*(S)$ stably finite.
If S is not a group then S is isomorphic to $E \times H \times E \sqcup \{0\}$,
where $E := E(S) \setminus \{0\}$ and H is a group.

Groupoid-TWW approach to going back

Goal: use algebraic properties of $S \rightsquigarrow$ prove quasi-diagonality.

There currently are two strategies:

(TWW) Nuclearity + UCT + faithful trace $\Rightarrow C_r^*(S)$ is QD

(Alt) Exploit the inner structure of S

Unfortunately, (TWW) does not quite work in this context:

Proposition (M. - 2021)

Suppose $G_{\mathcal{U}}(S)$ is minimal and $C_r^*(S)$ stably finite.

If S is not a group then S is isomorphic to $E \times H \times E \sqcup \{0\}$,

where $E := E(S) \setminus \{0\}$ and H is a group.

Remarks:

- Minimality is needed to guarantee faithfulness of the trace.
- Stable finiteness follows from quasi-diagonality.

Using $\mathcal{D}$-classes to prove quasi-diagonality

Theorem (M. - 2021)

Suppose that:

(1) No $\mathcal{D}$-class in S has infinitely many projections

(2) All the subgroups of S are amenable

Then $C_r^*(S)$ is quasi-diagonal.

Using $\mathcal{D}$-classes to prove quasi-diagonality

Theorem (M. - 2021)

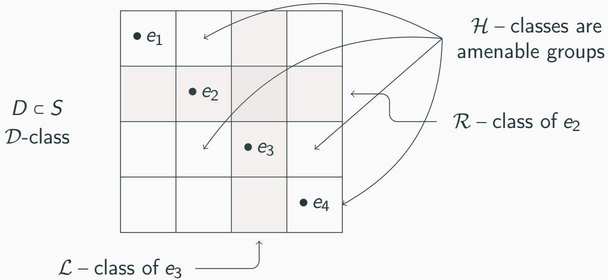
Suppose that:

(1) No $\mathcal{D}$-class in S has infinitely many projections

(2) All the subgroups of S are amenable

Then $C_r^*(S)$ is quasi-diagonal.

Very sketchy sketch of proof:



Using $\mathcal{D}$-classes to prove quasi-diagonality

Theorem (M. - 2021)

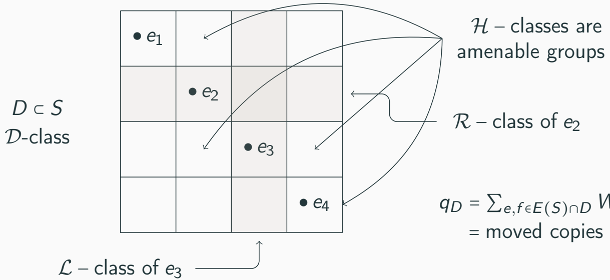
Suppose that:

(1) No $\mathcal{D}$-class in S has infinitely many projections

(2) All the subgroups of S are amenable

Then $C_r^*(S)$ is quasi-diagonal.

Very sketchy sketch of proof:



$$q_D = \textstyle\sum_{e,f \in E(S) \cap D} W_{r_e} V_{r_f} p_H V_{r_f}^* W_{r_e}^*$$
$$= \text{moved copies of } p_H$$

6. Summary and future work

Summary

Day's amenability:

- Split into dom.-measurable + localization $\rightsquigarrow$ *forward* dynamics.
- Dom.-measurable $\rightsquigarrow$ adequate Følner and paradoxicality.
- These measures characterize the (amenable) traces of $\mathcal{R}_S$.

Summary

Day's amenability:

- Split into dom.-measurable + localization $\rightsquigarrow$ *forward* dynamics.
- Dom.-measurable $\rightsquigarrow$ adequate Følner and paradoxicality.
- These measures characterize the (amenable) traces of $\mathcal{R}_S$.

Schützenberger graphs:

- Equipped S with a proper and right invariant metric
  $\rightsquigarrow$ distinguishes $\mathcal{L}$-classes, but does not implement $\geq$
- Viewed $\mathcal{R}_S$ as a crossed product and $C^*_u(\Lambda_S)$
  $\rightsquigarrow$ modulo finite labelings (automatic for groups)
- Studied Følner sets geometrically, and prop. A in $\mathcal{R}_S$.

Summary

Day's amenability:

- Split into dom.-measurable + localization $\rightsquigarrow$ *forward* dynamics.
- Dom.-measurable $\rightsquigarrow$ adequate Følner and paradoxicality.
- These measures characterize the (amenable) traces of $\mathcal{R}_S$.

Schützenberger graphs:

- Equipped S with a proper and right invariant metric
  $\rightsquigarrow$ distinguishes $\mathcal{L}$-classes, but does not implement $\geq$
- Viewed $\mathcal{R}_S$ as a crossed product and $C_u^*(\Lambda_S)$
  $\rightsquigarrow$ modulo finite labelings (automatic for groups)
- Studied Følner sets geometrically, and prop. A in $\mathcal{R}_S$.

Groupoids, quasi-diagonality and related notions:

- Related this study with groupoid amenability
- Upgraded 2-norm approx. $\rightsquigarrow$ norm-approx. (in some cases)

Future work

About metrics and property A:

- Unique proper and right invariant metric?
  Role of the finite labelings? Do these appear?
- Semigroups without property A $\rightsquigarrow$ *natural* examples?
- Reproduce non-A metric spaces with semigroups.

Future work

About metrics and property A:

- Unique proper and right invariant metric?
  Role of the finite labelings? Do these appear?
- Semigroups without property A $\rightsquigarrow$ *natural* examples?
- Reproduce non-A metric spaces with semigroups.

Quasi-diagonality and norm approximations:

- Characterize quasi-diagonality of $C_r^*(S)$ in terms of S.
  Role of the UCT? Traces? Nuclearity (or exactness)?
- Exploit decomposition into $\mathcal{D}$-classes and $\mathcal{H}$-classes.
- For groups, compute explicit quasi-diagonal approximation.

Future work

About metrics and property A:

- Unique proper and right invariant metric?
  Role of the finite labelings? Do these appear?
- Semigroups without property A $\rightsquigarrow$ *natural* examples?
- Reproduce non-A metric spaces with semigroups.

Quasi-diagonality and norm approximations:

- Characterize quasi-diagonality of $C_r^*(S)$ in terms of S.
  Role of the UCT? Traces? Nuclearity (or exactness)?
- Exploit decomposition into $\mathcal{D}$-classes and $\mathcal{H}$-classes.
- For groups, compute explicit quasi-diagonal approximation.

Interesting C*-algebras and their semigroup counterparts:

- Study the coarse geometry of AF-monoids. *Classifiable* ones?
- Inverse semigroup equivalent of $\mathcal{Z}$.

[1] F. Lledó and D. Martínez, *Notions of infinity in quantum physics*, 60 Years Alberto Ibort Fest Classical and Quantum Physics, Chapter 14, Springer (2019) (pp. 14)
https://doi.org/10.1007/978-3-030-24748-5_14

[2] P. Ara, F. Lledó and D. Martínez, *Amenability and paradoxicality in semigroups and C*-algebras*, J. Func. Anal. **279** (2020) 108530 (pp. 43)
https://doi.org/10.1016/j.jfa.2020.108530

[3] F. Lledó and D. Martínez, *The uniform Roe algebra of an inverse semigroup*, J. Math. Anal. Appl. **499** (2021) 124996 (pp. 28)
https://doi.org/10.1016/j.jmaa.2021.124996

[4] D. Martínez, *Quasi-diagonality for reduced inverse semigroup C*-algebras*, in preparation (2021)

Thank you for your attention. Questions?

uc3m Universidad Carlos III de Madrid

ICMAT INSTITUTO DE CIENCIAS MATEMÁTICAS

7. Some extra slides

Semigroups and Day's amenability

S semigroup $\rightsquigarrow$ binary and associative operation $(s,t) \mapsto st$

Key idea: multiplication is <u>not</u> injective... maybe $\nexists$ inverse s^{-1}

$$s^{-1}A := \{x \in S \mid sx \in A\} \rightsquigarrow ss^{-1}A \subsetneq A \subsetneq s^{-1}s\,A$$

Definition (Day 1957)

S is <u>*amenable*</u> if there is a prob. measure $\mu\colon \mathcal{P}(S) \to [0,1]$ such that $\mu\left(s^{-1}A\right) = \mu(A)$ for every $s \in S$ and $A \subset S$

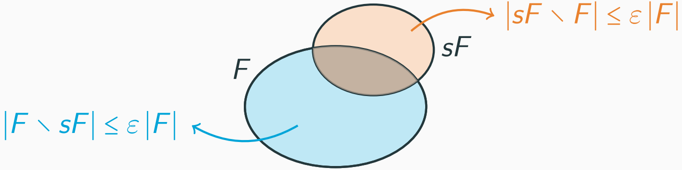
Examples:

- All amenable groups.
- $0 \in S \rightsquigarrow \mu(\{0\}) = 1$.
- $\mathbb{F}_2 \sqcup \{1\}$ is not amenable.
- $S := \{a, b\}$ where $ab = aa = a$ and $ba = bb = b$ is <u>not</u> amenable.

Questions: Følner condition? Paradoxical characterization?

Classical equivalences in semigroups

Note we have a choice to make:



- In general $|sF| \leq |F|$
- $|sF \setminus F| \leq |F \setminus sF|$, and equality when $|sF| = |F|$
- **Warning:** $|sF|$ small, and $sF \cap F = \varnothing \Rightarrow |sF \setminus F|$ small

(1) S satisfies the *Følner condition* if for every $\varepsilon > 0$ and $K \Subset S$ there is $F \Subset S$ such that $|sF \setminus F| \leq \varepsilon |F|$ for every $s \in K$

(2) S satisfies the *strong Følner condition* if for every $\varepsilon > 0$ and $K \Subset S$ there is $F \Subset S$ such that $|F \setminus sF| \leq \varepsilon |F|$ for every $s \in K$

Amenable, Følner and algebraic amenability

Theorem (many hands)

Let S be a semigroup. Consider:

(1) S has a strong Følner sequence
(2) S is amenable
(3) S has a Følner sequence
(4) $\mathbb{C}S$ is algebraically amenable

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4)

Proof: (1) $\Rightarrow$ (2): $\mu(A) = \lim_\omega |A \cap F_n| / |F_n|$

(2) $\Rightarrow$ (3): Namioka 1964, based on Følner 1957

(3) $\Rightarrow$ (4): $\{F_n\}_{n\in\mathbb{N}}$ is a Følner sequence then

$$\frac{\dim(s\,\mathbb{C}F_n + \mathbb{C}F_n)}{\dim(\mathbb{C}F_n)} \le \frac{\dim(\mathbb{C}(s\,F_n \cup F_n))}{\dim(\mathbb{C}F_n)} = \frac{|s\,F_n \cup F_n|}{|F_n|} \xrightarrow{n\to\infty} 1$$

Reversing the classical equivalences

Remark: none of the reverse implications hold.

(2) $\not\Rightarrow$ (1): Klawe 1977 $\rightsquigarrow$ right cancellative amenable semigroup
not strong Følner nor left cancellative

(3) $\not\Rightarrow$ (2): as $S = \{a, b\}$ is:

- Finite $\rightsquigarrow$ $F := S$ is a nice Følner set
- Not amenable: $\mu(\{a\}) = \mu(b^{-1}\{a\}) = 0 = \mu(a^{-1}\{b\}) = \mu(\{b\})$

(4) $\not\Rightarrow$ (3): a bit more involved...

Example (Ara, Lledó, M. - 2020)

Consider $\mathbb{F}_2^+ = \langle a, b \rangle \subset \mathbb{F}_2$, and

$$\alpha \colon \mathbb{F}_2^+ \to \mathrm{End}\,(\mathbb{N})\,, \quad \alpha_a\,(n) = \alpha_b\,(n) = \begin{cases} n-1 & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$

Then $\mathbb{N} \rtimes_\alpha \mathbb{F}_2^+$ is algebraically amenable but not Følner.

An argument for and against injectivity

All of the above differences boil down to $|F| \neq |sF|$

Definition: S is <u>*(left) cancellative*</u> if $sx = sy$ then $x = y$

Nice behavior with cancellation:

- $A = s^{-1}sA$
- $|F| = |sF|$, and hence
- Strong Følner $\Leftrightarrow$ amenable $\Leftrightarrow$ Følner

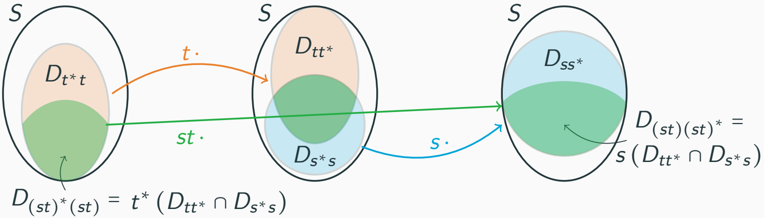
Naughty behavior with cancellation:

- $ss^{-1}A \subsetneq A$
- Not a clear candidate for *paradoxicality*, as
  $\nexists\, t \in S$ such that $t^{-1}s^{-1}A = A \rightsquigarrow S$ lacks the inverse of s

Solution: having sets $D_{s,i} \subset S$ such that $s\colon D_{s,1} \to D_{s,2}$ is bijective,
and do everything up to these *domains*

Recall that the <u>*Wagner-Preston*</u> representation $v\colon S \to \mathcal{I}(S)$ is

- $\underline{D_{s^*s} := \{x \in S \mid x = s^*sx\}} = s^*s \cdot S$ is the *domain of* s^*s
- $s\colon D_{s^*s} \to D_{ss^*}$, where $x \mapsto sx$



This idea can be taken a bit further:

Definition (Sieben - 1997)

A <u>*representation*</u> of S on the discrete set X is a homomorphism $\alpha\colon S \to \mathcal{I}(X)$, where $s \mapsto (\alpha_s\colon D_{s^*s} \to D_{ss^*})$

Representations - and one example to contain them all

Recall: $\mathcal{I}(X) = \{(t, A, B) \mid A, B \subset X \text{ and } t\colon A \mapsto B \text{ is a bijection}\}$

- In particular $\mathcal{I}(\mathbb{N})$ contains every countable inverse semigroup
- $\frac{\text{bijections}}{\text{groups}} = \frac{\text{partial bijections}}{\text{inv. semigroups}}$
- Idempotents $e \in E(S) \rightsquigarrow$ subsets of X

Remark: recall that if $S = G$ is a group then

$$\begin{aligned}\{\text{actions } \alpha\colon G \to \mathrm{Perm}(X)\} &\leftrightarrow \{\text{congruences in } G\} \\ &\leftrightarrow \{\text{normal subgroups } N \subset G\}\end{aligned}$$

However, $\alpha\colon S \to \mathcal{I}(X)$ induces only a congruence on S

Warning: no clear notion of *normal sub-semigroup*

$\rightsquigarrow$ no obvious description of $\{\text{congruences on } S\}$

The type semigroup

Given $\alpha : S \to \mathcal{I}(X)$ consider the commutative monoid

$$\mathrm{Typ}(\alpha) := \langle [A] \text{ where } A \subset X \rangle \text{ subject to}$$

(i) $[\varnothing] = 0$

(ii) $[A] = [\alpha_s(A)]$ whenever $A \subset D_{s^*s}$

(iii) $[A \cup B] = [A] + [B]$ if $A \cap B = \varnothing$

Lastly, $\alpha, \beta \in \mathrm{Typ}(\alpha)$ then $\underline{\alpha \leq \beta}$ if $\alpha + \gamma = \beta$ for some $\gamma \in \mathrm{Typ}(\alpha)$

Lemma (Ara, Lledó, M. - 2020)

Let $\mathrm{Typ}(\alpha)$ be as above, and let $[A], [B] \in \mathrm{Typ}(\alpha)$. Then

(1) If $[A] \leq [B]$ and $[B] \leq [A]$ then $[A] = [B]$

(2) If $n \cdot [A] = n \cdot [B]$ for any $n \in \mathbb{N}$ then $[A] = [B]$

(3) If $(n+1) \cdot [A] \leq n \cdot [A]$ for some $n \in \mathbb{N}$ then $[A] = 2 \cdot [A]$

Classical equivalences renovated I

Theorem (Ara, Lledó, M. - 2020)

Let $\alpha\colon S \to \mathcal{I}(X)$ be an inverse semigroup. TFAE:

(1) X is domain-measurable.
(2) X is not paradoxical.
(3) X is domain-Følner.

(1) $\Rightarrow$ (3): follow the algorithm:

- $m(p_B) := \mu(B)$ + extend by lineary and continuity
- $m \in (\ell^\infty(X))^*$ is a normalized functional
- Recall: $\{\text{normal states}\} \subset (\ell^\infty(X))^*$ are dense (weak-*)

Approximating m by normal states, and applying Namioka's trick

$$\rightsquigarrow \; h_\lambda \in \ell^1(X)_1^+ \quad \text{such that} \quad \|s^* h_\lambda - s^* s h_\lambda\|_1 \to 0$$

Some *inner level set* of h_λ is a good enough Følner set.

Classical equivalences renovated II

(1) ⇒ (2): if it was paradoxical then

$$2 = \sum_{i=1}^{n} \mu(s_i A_i) + \sum_{j=1}^{m} \mu\left(t_j B_j\right) = \sum_{i=1}^{n} \mu(A_i) + \sum_{j=1}^{m} \mu\left(B_j\right)$$
$$= \mu(A_1 \sqcup \cdots \sqcup A_n \sqcup B_1 \sqcup \cdots \sqcup B_m) \leq \mu(X) = 1$$

(2) ⇒ (1): note $2 \cdot [X] \nleq [X]$ in $\mathrm{Typ}(\alpha)$, and by the Lemma

$$(n+1) \cdot [X] \nleq n \cdot [X] \text{ for any } n \in \mathbb{N}$$

Theorem (Tarski - 1929)

Let $(\mathcal{T}, +)$ be a commutative monoid, with neutral element and $\epsilon \in \mathcal{T}$. The following are equivalent:

(a) $(n+1) \cdot \epsilon \nleq n \cdot \epsilon$ for any $n \in \mathbb{N}$
(b) There is a homo. $\nu\colon \mathcal{T} \to [0, \infty]$ with $\nu(\epsilon) = 1$

Hence there is a homomorphism $\nu\colon \mathrm{Typ}(\alpha) \to [0, \infty]$ with $\nu([X]) = 1$. Then $\mu(B) := \nu([B])$ is a domain-measure.

The localization condition localized in time

The localization already appeared in previous work:

Theorem (Gray, Kambites - 2017)

Let S be a semigroup satisfying the *Klawe condition*. Then

S is amenable $\Leftrightarrow$ there are $F_n \subset S$ such that

$$|sF_n \setminus F_n| \,/\, |F_n| \to 0 \quad \text{and} \quad |sF_n| = |F_n|$$

F_n can be found where s acts injectively $\rightsquigarrow D_{s^*s}$

Indeed: if $\alpha = v\colon S \to \mathcal{I}(S)$, and S is amenable when F_n is eventually within D_{s^*s} $\rightsquigarrow$ $F_n \subset D_{s^*s} \Rightarrow |F_n| = |sF_n|$

Wagner-Preston case, or when X = S

Before: what happens when $X = S$ and $\alpha = v\colon S \to \mathcal{I}(S)$?

Consider the *action* of S given by:

- $E_{s^*s} = \{f \in \ell^\infty(S) \mid \operatorname{supp}(f) \subset D_{s^*s}\}$
- For any $f \in E_{s^*s}$ consider $(sf)(x) := f(s^*x)$ when $x \in D_{ss^*}$

And construct $\ell^\infty(S) \rtimes_r S \subset \mathcal{B}\left(\ell^2 S \otimes \ell^2 S\right)$

Proposition (Ara, Lledó, M. - 2020)

With the above notation: $\mathcal{R}_S \cong \ell^\infty(S) \rtimes_r S$

Proof: check that $U(\ell^\infty(S) \rtimes_r S)\, U^* \cong 1 \otimes \mathcal{R}_S \cong \mathcal{R}_s$, where

$$U\colon \ell^2 S \otimes \ell^2 S \to \ell^2 S \otimes \ell^2 S,\ \ \delta_x \otimes \delta_y \mapsto \begin{cases} \delta_x \otimes \delta_{yx} & \text{if } xx^* = y^*y \\ 0 & \text{otherwise} \end{cases}$$

All the traces of $\mathcal{R}_S$ are amenable

Theorem (Ara, Lledó, M. - 2020)

There is a bijection between the domain-measures on X and the traces of $\mathcal{R}_X$. In particular, all the traces of $\mathcal{R}_X$ are amenable.

Proof: the map $\mu \mapsto \tau : \mathcal{B}\left(\ell^2 X\right) \xrightarrow{\mathcal{E}} \ell^\infty(X) \xrightarrow{m} \mathbb{C}$ is a bijection:

→ τ is a trace (see above Theorem)

→ The map is bijective since:

Lemma (Ara, Lledó, M. - 2020)

All traces of $\mathcal{R}_S$ factor under $\mathcal{E}$, and hence $m = \mathcal{E} \circ \tau$ is a mean.

Key point: if $sx \neq x$ for all $x \in A \subset D_{s^*s}$ it can be 3-colored, i.e.,

$A = B_1 \sqcup B_2 \sqcup B_3$ where $sB_i \cap B_i = \emptyset$. Hence for every trace φ

$$\begin{aligned}\varphi(V_s p_A) &= \varphi(V_s p_{B_1}) + \varphi(V_s p_{B_2}) + \varphi(V_s p_{B_3}) \\ &= \varphi(p_{B_1} V_s p_{B_1}) + \varphi(p_{B_2} V_s p_{B_2}) + \varphi(p_{B_3} V_s p_{B_3}) = 0\end{aligned}$$

Property A and nuclearity of $\mathcal{R}_S$

Theorem (Lledó, M. - 2021)

If Λ_S has property A (as a graph) then $\mathcal{R}_S$ is nuclear.

Note: this is based on a similar result for $\mathcal{L}$-classes:

Proof: given a $\xi : X \to \ell^1(X)_1^+$ and $c > 0$ consider

$$\varphi : \mathcal{R}_S \to \prod_{x \in S} M_{B_c(x)} \subset \ell^\infty(S) \otimes M_k, \text{ where } a \mapsto \left(p_{B_c(x)}\, a\, p_{B_c(x)}\right)_{x \in S}$$

and

$$\psi : \ell^\infty(S) \otimes M_k \to \mathcal{R}_S, \text{ where } (b_x)_{x \in S} \mapsto \sum_{x \in S} \xi_x^* \, b_x \xi_x$$

with $\xi_x \delta_y := \xi_y(x)\, \delta_y$ gives a c.p.c. approximation of $\mathcal{R}_S$:

- Both φ and ψ are completely positive
- $\|fV_s - \psi(\varphi(fV_s))\| \leq \|fV_s\| \cdot \sup_{y \in D_{s^*s}} |1 - \langle \xi_{sy}, \xi_y \rangle| \leq \varepsilon$

E-unitary semigroups and relation with property A

Recall: $G(S) = S/\sigma$, where $s\sigma t$ iff $se = te$ for some $e \in E(S)$

- Known as the *maximal homomorphic image of S*
- $G(S)$ is always a group, and let $\sigma\colon S \to G(S)$ the quotient map

Theorem (Duncan and Namioka - 1978)

S is amenable if, and only if, $G(S)$ is amenable.

However: $G(S)$ loses information of $S \rightsquigarrow$ might even $G(S) = \{1\}$

Definition (classical)

S is *E-unitary* if $\sigma\colon S \to G(S)$ is injective in every $\mathcal{L}$-class.

Theorem (Anantharaman-Delaroche - 2016)

Let S be E-unitary. $C_r^*(S)$ is exact $\Leftrightarrow$ $G(S)$ is exact.

E-unitary semigroups and relation with property A

Goal: prove Anantharaman-Delaroche's result geometrically

Theorem (Lledó, M. - 2021)

Let $S = \langle K \rangle$ be E-unitary and (S, K) admit a finite labeling.

$$\Lambda_S \text{ has property A} \Leftrightarrow G(S) \text{ has property A.}$$

(Recall that, by Ozawa, property A = exactness for groups)

Proof $\Rightarrow$**:** given $\xi\colon S \to \ell^1(S)^1_+$ for S let

$$\zeta\colon G(S) \to \ell^1(G(S)), \text{ where } \zeta_{\sigma(s)}(\sigma(t)) := \lim_{n\to\omega} \xi_{se_n}(te_n)$$

for some $e_1 \geq \cdots \geq e_n \geq \cdots \in E(S)$ is *eventually below everything*

- Locally $G(S)$ is Λ_{L_e} for e sufficiently small
- Hence, the same approx. for Λ_S does the trick for $G(S)$

Proof $\Leftarrow$**:** only known via C*-arguments

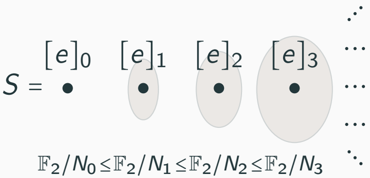
Higson-Lafforgue-Skandalis and Willett's example

Example: box space without property A

- $\mathbb{F}_2 = \langle a, b \mid - \rangle$ free non-abelian group on 2 elements
- Let $\{N_k\}_{k\in\mathbb{N}}$ be a descending sequence of normal subgroups of finite index such that $\cap_{k\in\mathbb{N}} N_k = \{1\}$
- $S := \sqcup_{k\in\mathbb{N}} \mathbb{F}_2/N_k$, where $[g]_i \cdot [h]_j := [gh]_{\min\{i,j\}}$.

Remark:

(1) S is amenable $\rightsquigarrow$ as it has a 0
(2) S does not have property A $\rightsquigarrow$ $\mathcal{L}$-classes are *complicated*
(3) S does not admit a finite labeling $\rightsquigarrow$ same as $(\mathbb{N}, \min)$



Paterson's universal groupoid

Recall the construction of <u>*Paterson's universal groupoid*</u> $G_{\mathcal{U}}(S)$:

- Unit space $G_{\mathcal{U}}(S)^{(0)} := \{\xi \subset E(S) \text{ filter}\} \subset \{0,1\}^{E(S)}$
- Action of S: $\theta_s \colon D^{\theta}_{s^*s} \to D^{\theta}_{ss^*}$, $\theta_s(\xi) = \{e \geq sfs^*\}_{f \in \xi}$
- Groupoid $G_{\mathcal{U}}(S) = \left\{[s,\xi] \text{ where } s \in S \text{ and } \xi \in D^{\theta}_{s^*s}\right\}$
  Subject to some germ-like equivalence relation...
- Operation: $[t, \theta_s(\xi)] \cdot [s,\xi] = [ts,\xi]$

Example: $\mathcal{B} = \langle a, a^* \mid a^*a = 1\rangle$ then

- $G_{\mathcal{U}}(\mathcal{B})^{(0)} = E(S) \sqcup \{\infty\} = \mathbb{N} \sqcup \{\infty\}$
- $G_{\mathcal{U}}(\mathcal{B})_k = \{[s,k]\}_{s^*s \geq a^k a^{*k}} = \left\{\left[a^i a^{*j}, k\right]\right\}_{j \leq k}$
- $G_{\mathcal{U}}(\mathcal{B})_\infty = \mathbb{Z}$

Groupoid amenability, nuclearity and weak containment

Theorem (Lance '70s, for groups!)

G is amenable $\Leftrightarrow$ $C_r^*(G)$ is nuclear $\Leftrightarrow$ $C_r^*(G) = C^*(G)$

This has become a driving force of groupoid amenability:

Definition (Renault - 1980)

G (étale) is *amenable* if for any $\varepsilon > 0$ and compact $K \subset G$ there is a cont. compc. supported $\eta \colon G \to [0,1]$ such that for all $s \in K$

$$\left|1 - \sum_{s(h)=r(g)} \eta(h)\right| \leq \varepsilon \quad \text{and} \quad \sum_{s(h)=r(g)} |\eta(h) - \eta(hg)| \leq \varepsilon$$

Question:

- Relation between $G_{\mathcal{U}}(S)$ amenable and S amenable?
- How does $G_{\mathcal{U}}(S)$ enter the picture?

Groupoid amenability vs domain-measurability

Question: relation between $G_{\mathcal{U}}(S)$ amenable and S amenable?

Short answer: no relation, as

(1) S amenable/domain-measurable $\rightsquigarrow$ traces in $\mathcal{R}_S$
(2) $G_{\mathcal{U}}(S)$ amenable $\rightsquigarrow$ nuclearity of $C_r^*(S)$ $\rightsquigarrow$ nuclearity of $\mathcal{R}_S$

Examples:

(i) $\mathbb{F}_2 \sqcup \{0\}$ is amenable but non-nuclear reduced C*-algebra
(Can also be done with a *box space* with a prop. (T) group)
(ii) $C_r^*(\mathbb{F}_2 \sqcup \{0\}) \neq C^*(\mathbb{F}_2 \sqcup \{0\})$
(iii) Nica gave a non-amenable semigroup with nuclear C*-algebra